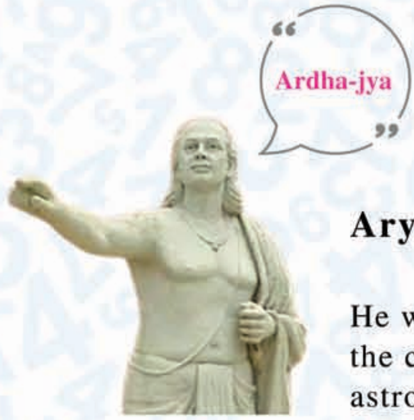
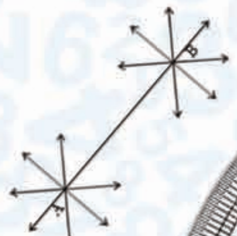
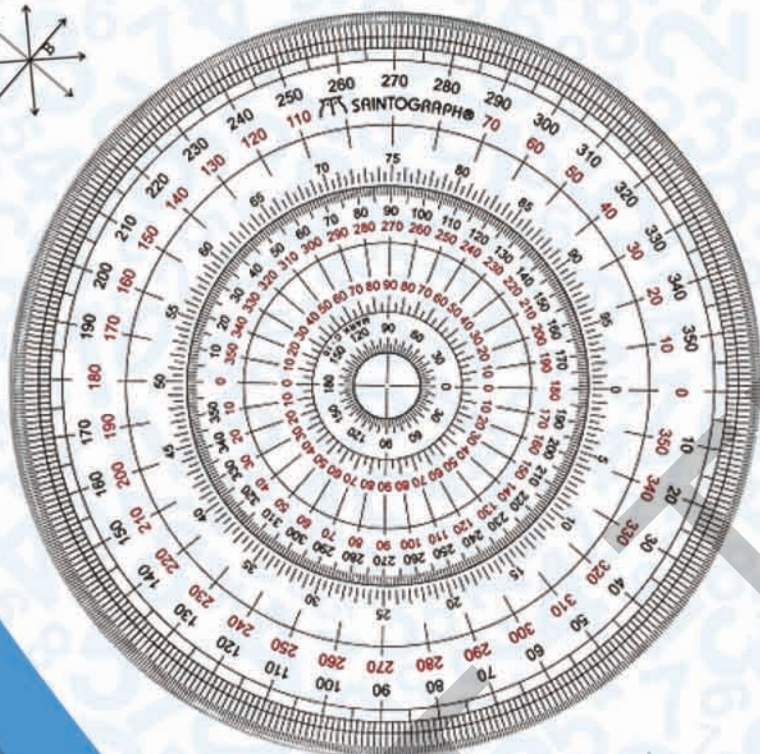
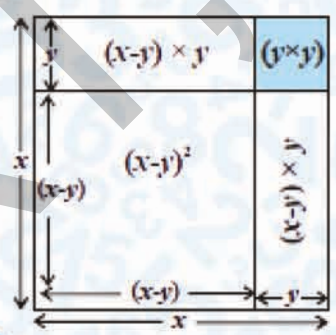


Aryabhata (476–550 CE)

He was an Indian mathematician and astronomer of the classical age of Indian mathematics and Indian astronomy.





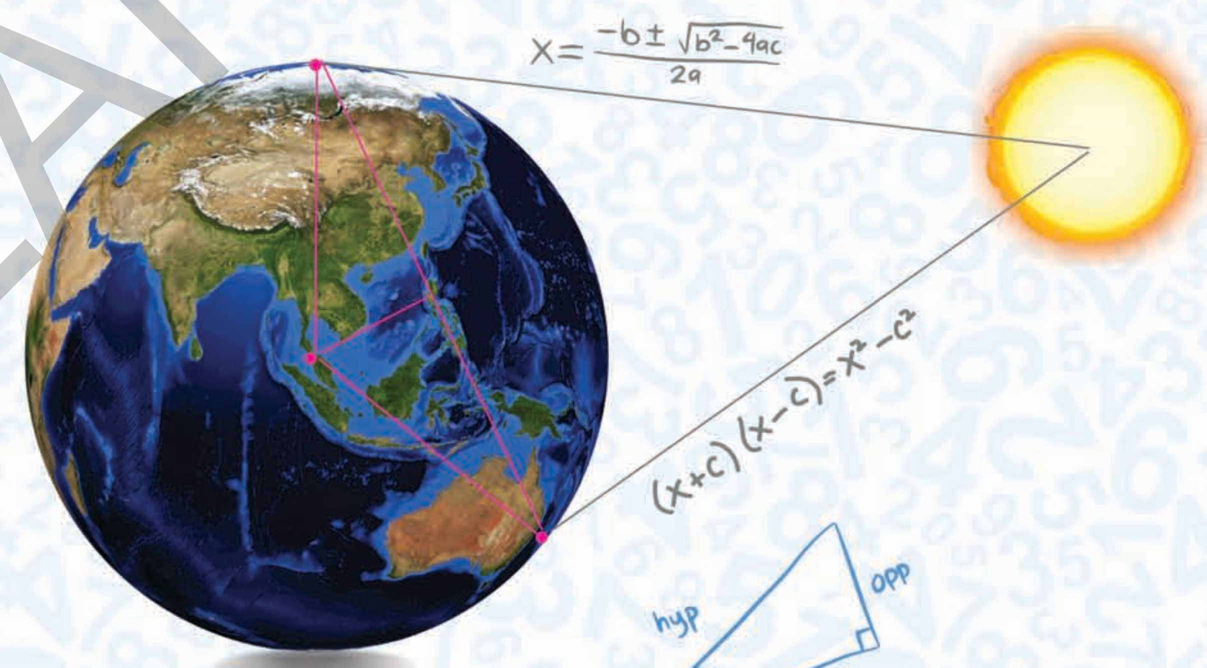


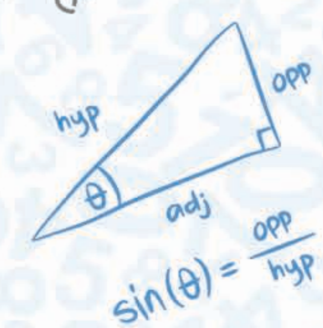
MATHEMATICS

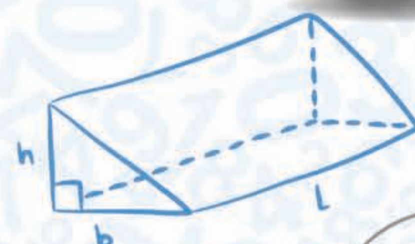
CLASS XI

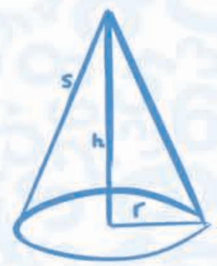
MATHEMATICS

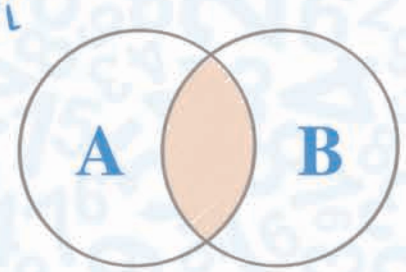
CLASS 9











State Council of Educational Research and Training
Telangana, Hyderabad

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MATHEMATICS

CLASS - IX

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Respect the Law **Grow by Education**

Get the Rights **Behave Humbly**

First Published 2013

New Impressions 2014, 2015, 2016, 2017, 2018, 2019, 2020, 2021, 2022,2023

This Book has been printed on 90 G.S.M. Maplitho
Title Page 250 G.S.M. White Art Card

Government's Gift for Students' Progress 2023-24

విద్యార్థుల వికాసానికి ప్రభుత్వ కానుక 2023-24

Printed in India
at the Telangana Govt. Text Book Press,
Mint Compound, Hyderabad,
Telangana.

Foreword

Education is a process of human enlightenment and empowerment. Recognizing the enormous potential of education, all progressive societies have committed themselves to the Universalization of Elementary Education with an explicit aim of providing quality education to all. As the next step, universalization of Secondary Education has gained momentum.

The secondary stage marks the beginning of the transition from functional mathematics studied upto the upper primary stage to the study of mathematics as a discipline. The logical proofs of propositions, theorems etc. are introduced at this stage. Apart from being a specific subject, it is to be treated as a concommitant to any subject involving analysis as reasoning.

I am confident that the children in our state of Telangana learn to enjoy mathematics, make mathematics a part of their life experience, pose and solve meaningful problems, understand the basic structure of mathematics by reading this text book.

For teachers, to understand and absorb critical issues on curricular and pedagogic perspectives duly focusing on learning in place of marks, is the need of the hour. Also coping with a mixed class room environment is essentially required for effective transaction of curriculum in teaching learning process. Nurturing class room culture to inculcate positive interest among children with difference in opinions and presumptions of life style, to infuse life into knowledge is a thrust in the teaching job.

With an intention to help the students to improve their understanding skills in both the languages i.e. English and Telugu, the Government of Telangana has redesigned this book as bilingual textbook in two parts. Part-1 comprises 1, 2, 3, 4,5, 6, 7 chapters and Part-2 comprises 8, 9, 10, 11, 12, 13, 14, 15 chapters.

The afore said vision of mathematics teaching presented in State Curriculum Frame work (SCF -2011) has been elaborated in its mathematics position paper which also clearly lays down the academic standards of mathematics teaching in the state. The text books make an attempt to concretize all the sentiments.

The State Council for Education Research and Training Telangana appreciates the hard work of the text book development committee and several teachers from all over the state who have contributed to the development of this text book. Our special thanks to Faculty of School of Education Tata Institute of Social Sciences (TISS), Hyderabad and Sri Ramesh Khade, Communication Officer, CETE, TISS-Mumbai and Designers identified by SCERT for their technical support in redesigning of the textbooks.

Place : Hyderabad — Director

Date : 07 December 2012 — SCERT, Hyderabad

NATIONAL ANTHEM

Jana-gana-mana-adhinayaka, jaya he
Bharata-bhagya-vidhata.
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchhala-jaladhi-taranga.
Tava shubha name jage,
Tava shubha asisa mage,
Gahe tava jaya gatha,
Jana-gana-mangala-dayaka jaya he
Bharata-bhagya-vidhata.
Jaya he! jaya he! jaya he!
Jaya jaya jaya, jaya he!!

- *Rabindranath Tagore*

PLEDGE

"India is my country. All Indians are my brothers and sisters.
I love my country, and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall give my parents, teachers and all elders respect,
and treat everyone with courtesy. I shall be kind to animals
To my country and my people, I pledge my devotion.
In their well-being and prosperity alone lies my happiness."

- *Pydimarri Venkata Subba Rao*

INDEX

THE CONSTITUTION OF INDIA

Preamble

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC and to secure to all its citizens:

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the unity and integrity of the Nation;

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.

Chapter 1

Real Numbers

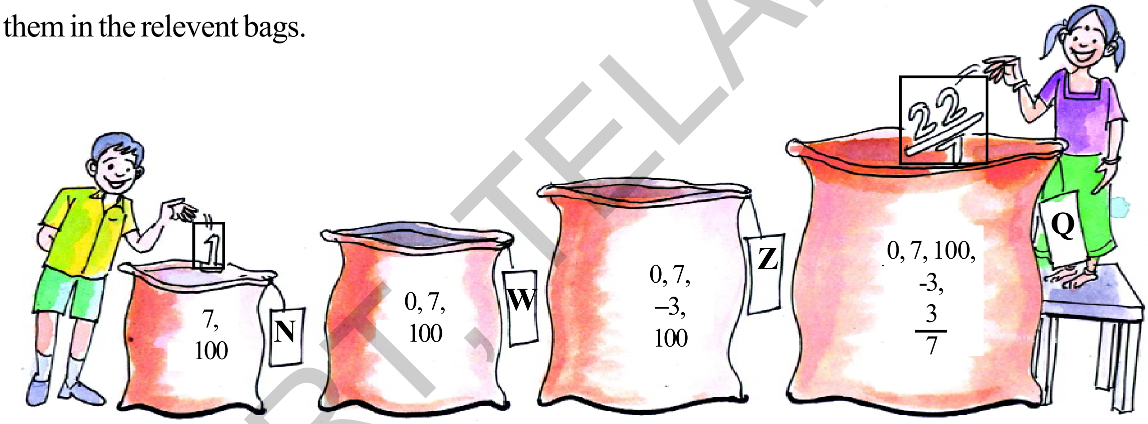
1.1 Introduction

Let us have a brief review of various types of numbers.

Observe the following numbers.

$7, 100, 9, 11, -3, 0, -\frac{1}{4}, 5, 1, \frac{3}{7}, -1, 0.12, -\frac{13}{17}, 13.222 \ldots, 19, \frac{-5}{3}, \frac{213}{4}, \frac{-69}{1}, \frac{22}{7}, 5.\overline{6}$

John and Sneha want to label the above numbers and put them in the bags they belong to. Some of the numbers are in their respective bags..... Now you pick up rest of the numbers and put them into the bags to which they belong. If a number can go in more than one bag then copy the number and put them in the relevent bags.



You have observed bag N contains natural numbers. Bag W contains whole numbers. Bag Z contains integers and bag Q contains rational numbers.

We observe that the bag Z contains integers which is the collection of negative numbers and whole numbers. It is denoted by I or Z and we write,

$$Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

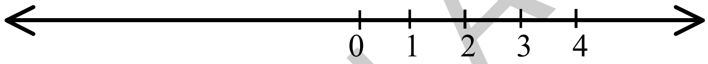
Similarly the bag Q contains all numbers that are of the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

Further, we can observe that all the numbers that are of the bag N, the bag W and the bag Z are present in the bag Q.

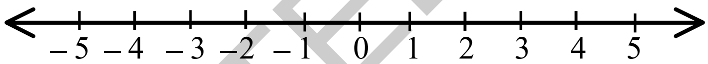
You might have noticed that natural numbers, whole numbers, integers and rational numbers can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

For example, -15 can be written as $\frac{-15}{1}$; here p = -15 and q = 1. Observe the Example $\frac{1}{2}=\frac{2}{4}=\frac{10}{20}=\frac{50}{100}$ ... and so on. These are equivalent rational numbers . It means that the rational numbers do not have a unique representation in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. However, when we say $\frac{p}{q}$ is a rational number or when we represent $\frac{p}{q}$ on a number line, we assume that $q \neq 0$ and that p and q have no common factors other than the universal factor '1' (i.e., p and q are co-primes.) There are infinitely many rational numbers equivalent to $\frac{1}{2}$, we will choose $\frac{1}{2}$ i.e., the simplest form to represent all of them.

You know that how to represent whole numbers on the number line. We draw a line and mark a point '0' on it. Then we place the numbers in equal distances on the right side of the point '0' and label the points of division as 1, 2, 3, 4, …



Similarly we represent integers on number line as.



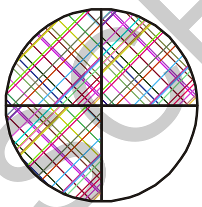
Do you remember how to represent the rational numbers on a number line?

To recall this, let's take the rational number $\frac{3}{4}$ and represent it on number line as well as pictorially.

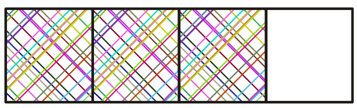
We know that in $\frac{3}{4}$, 3 is the numerator and 4 is the denominator.

Which means, 3 parts are taken out of 4 equal parts from a given unit.

Here are few representations of $\frac{3}{4}$.

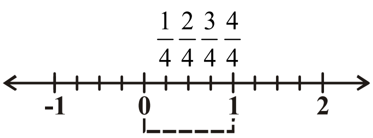


$\frac{3}{4}$



$\frac{3}{4}$

(Pictorially)



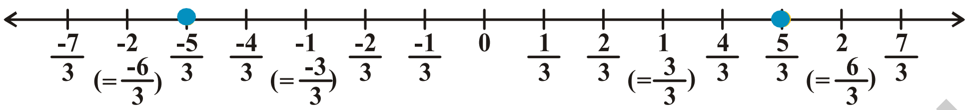
(Number line)

Think & Discuss

Can we represent all rational numbers pictorially?

Example-1. Represent $\frac{5}{3}$ and $-\frac{5}{3}$ on the number line.

Solution : Draw an integer line representing $-2, -1, 0, 1, 2$.



Divide each unit into three equal parts to the right and left sides of zero respectively. Take five parts out of these. The fifth point on the right of zero represents $\frac{5}{3}$ and the fifth one to the left of zero represents $\frac{-5}{3}$.

Do This

1. Represent $\frac{-3}{4}$ on the number line.
2. Write 0, 7, 10, -4 in $\frac{p}{q}$ form.
3. **Guess my number** : Your friend chooses an integer between 0 and 100. You have to find out that number by asking questions, but your friend can answer only in 'yes' or 'no'. What strategy would you use?

Example-2. Are the following statements True? Give reasons for your answers with an example.

i. Every rational number is an integer.

ii. Every integer is a rational number

iii. Zero is a rational number

Solution : i. False: For example, $\frac{7}{8}$ is a rational number but not an integer.

ii. True: Because any integer can be expressed in the form $\frac{p}{q}$ $(q \neq 0)$ for example $-2 = \frac{-2}{1} = \frac{-4}{2}$. Thus it is a rational number.

(i.e. any integer 'b' can be represented as $\frac{b}{1}$)

iii. True: Because 0 can be expressed as $\frac{0}{2}, \frac{0}{7}, \frac{0}{13}$ ($\frac{p}{q}$ form, where p, q are integers and $q \neq 0$)

('0' can be represented as $\frac{0}{x}$ where 'x' is an integer and $x \neq 0$)

Example-3. Find two rational numbers between 3 and 4 by mean method.

Solution :

Method-I : We know that the rational number that lies between two rational numbers a and b can be found using mean method i.e. $\frac{a+b}{2}$.

Here $a = 3$ and $b = 4$, (we know that $\frac{a+b}{2}$ is the mean of two integers 'a', 'b' and it lies between 'a' and 'b')

So, $\frac{(3+4)}{2} = \frac{7}{2}$ which is in between 3 and 4. $3 < \frac{7}{2} < 4$

If we continue the above process, we can find many more rational numbers between 3 and 4

$$\frac{3+\frac{7}{2}}{2} = \frac{\frac{6+7}{2}}{2} = \frac{\frac{13}{2}}{2} = \frac{13}{2\times 2} = \frac{13}{4}$$

$$3 < \frac{13}{4} < \frac{7}{2} < 4$$

Method-II : The other option to find two rational numbers in single step.
Since we want two numbers, we write 3 and 4 as rational numbers with denominator $2 + 1 = 3$

i.e., $3 = \frac{3}{1} = \frac{6}{2} = \frac{9}{3} = \frac{12}{4}$ and $4 = \frac{4}{1} = \frac{8}{2} = \frac{12}{3} = \frac{16}{4}$

Then you can see that $\frac{10}{3}, \frac{11}{3}$ are rational numbers between 3 and 4.

$$3 = \frac{9}{3} < \left(\frac{10}{3} < \frac{11}{3}\right) < \frac{12}{3} = 4$$

Now if you want to find 5 rational numbers between 3 and 4, then we write 3 and 4 as rational number with denominator $5 + 1 = 6$.

i.e. $3 = \frac{18}{6}$ and $4 = \frac{24}{6}$ $\qquad 3 = \frac{18}{6} < \left(\frac{19}{6}, \frac{20}{6}, \frac{21}{6}, \frac{22}{6}, \frac{23}{6}\right) < \frac{24}{6} = 4$

From this, you might have realised the fact that there are infinitely many rational numbers between 3 and 4. Check, whether this holds good for any other two rational numbers? Thus we can say that , there exist infinite number of rational numbers between any two given rational numbers.

Do This

i. Find any five rational numbers between 2 and 3 using mean method.

ii. Find any 10 rational numbers between $-\frac{3}{11}$ and $\frac{8}{11}$.

Example-4. Write $\frac{7}{16}, \frac{10}{7}$ and $\frac{2}{3}$ in decimal form.

Solution :

$$\begin{array}{r} 0.4375 \\ 16\overline{)7.00000} \\ 0 \\ \hline 70 \\ 64 \\ \hline 60 \\ 48 \\ \hline 120 \\ 112 \\ \hline 80 \\ 80 \\ \hline 0 \end{array} \qquad \begin{array}{r} 1.428571 \\ 7\overline{)10} \\ 7 \\ \hline 30 \\ 28 \\ \hline 20 \\ 14 \\ \hline 60 \\ 56 \\ \hline 40 \\ 35 \\ \hline 50 \\ 49 \\ \hline 10 \\ 7 \\ \hline 3 \end{array} \qquad \begin{array}{r} 0.666 \\ 3\overline{)2.0000} \\ 18 \\ \hline 20 \\ 18 \\ \hline 20 \\ 18 \\ \hline 2 \end{array}$$

$\therefore \frac{7}{16} = 0.4375$

is a terminating decimal

$\therefore \frac{10}{7} = 1.\overline{428571}$

is a non-terminating recurring decimal

$\therefore \frac{2}{3} = 0.666 = 0.\overline{6}$

is a non-terminating recurring decimal

From above examples, we notice that every rational number can be expressed as a terminating decimal or a non terminating recurring decimal.

Do This

Write (i) $\frac{1}{17}$ (ii) $\frac{1}{19}$ in decimal form.

Example -5. Express 3.28 in the form of $\frac{p}{q}$ (where p and q are integers, $q \neq 0$).

Solution : $3.28 = \frac{328}{100}$

$$= \frac{328 \div 2}{100 \div 2} = \frac{164}{50}$$

$$= \frac{164 \div 2}{50 \div 2} = \frac{82}{25}$$ (Numerator and denominator are co-primes)

$$\therefore 3.28 = \frac{82}{25}$$

Example-6. Express $1.\overline{62}$ in $\frac{p}{q}$ form where $q \neq 0$; p, q are integers.

Solutions : Let $x = 1.626262.....$ (1)

multiplying both sides of equation (1) by 100, we get

$100x = 162.6262...$ (2)

Subtracting (2) from (1) we get

$$\begin{array}{rl} 100x = & 162.6262... \\ x = & 1.6262... \\ - & - \\ \hline 99x = & 161 \\ \hline \end{array}$$

$$x = \frac{161}{99}$$

$$\therefore 1.\overline{62} = \frac{161}{99}$$

Try This

I. Find the decimal values of the following:

i. $\frac{1}{2}$ ii. $\frac{1}{2^2}$ iii. $\frac{1}{5}$ iv. $\frac{1}{5 \times 2}$

v. $\frac{3}{10}$ vi. $\frac{27}{25}$ vii. $\frac{1}{3}$ viii. $\frac{7}{6}$

ix. $\frac{5}{12}$ x. $\frac{1}{7}$

Observe the following decimals

$\frac{1}{2} = 0.5$ $\frac{1}{10} = 0.1$ $\frac{32}{5} = 6.4$ $\frac{1}{3} = 0.333...$ $\frac{4}{15} = 0.2\overline{6}$

Can you guess the character of the denominator of a fraction which can be in the form of terminating decimal?

Write prime factors of denominator of each rational number.

What did you observe from the results?

We observe that a rational number is a terminating decimal, only when the prime factors of denominator are 2, 5 only i.e. if the denominator is expressed as $2^m . 5^n$, where m and n are non-negative integers.

Exercise - 1.1

1. (a) Write any three rational numbers

   (b) Explain rational number in your own words.

2. Give one example each to the following statements.

   i. A number which is rational but not an integer

   ii. A whole number which is not a natural number

   iii. An integer which is not a whole number

   iv. A number which is natural number, whole number, integer and rational number.

   v. A number which is an integer but not a natural number.

3. Find five rational numbers between 1 and 2.

4. Insert three rational numbers between $\frac{3}{5}$ and $\frac{2}{3}$

5. Represent $\frac{8}{5}$ and $\frac{-8}{5}$ on the number line

6. Express the following rational numbers in decimal form.

   I. i) $\frac{242}{1000}$ ii) $\frac{354}{500}$ iii) $\frac{2}{5}$ iv) $\frac{115}{4}$

   II. i) $\frac{2}{3}$ ii) $-\frac{25}{36}$ iii) $\frac{22}{7}$ iv) $\frac{11}{9}$

7. Express each of the following decimals in $\frac{p}{q}$ form (where $q \neq 0$ and p, q are integers)

   i) 0.36 ii) 15.4 iii) 10.25 iv) 3.25

8. Express each of the following decimal numbers in $\frac{p}{q}$ form

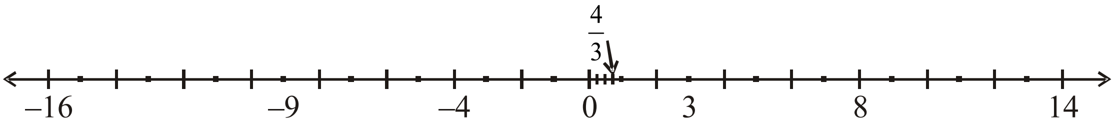
   i) $0.\overline{5}$ ii) $3.\overline{8}$ iii) $0.\overline{36}$ iv) $3.12\overline{7}$

9. Without actually dividing find which of the following are terminating decimals.

   (i) $\frac{3}{25}$ (ii) $\frac{11}{18}$ (iii) $\frac{13}{20}$ (iv) $\frac{41}{42}$

1.2 Irrational Numbers

Let us take a look at the number line again. Are we able to represent all the numbers on the number line? The fact is that there are infinite numbers left on the number line. Now let us see those numbers.



To understand this, consider these equations

(i) $x^2 = 4$ (ii) $3x = 4$ (iii) $x^2 = 2$

For equation (i) we know that value of x for this equation are 2 and –2. We can plot 2 and –2 on the number line.

For equation (ii) $3x = 4$ on dividing both sides by, 3 we get $\frac{3x}{3} = \frac{4}{3} \Rightarrow x = \frac{4}{3}$. We can plot this on the number line.

When we solve the equation (iii) $x^2 = 2$, taking square root on both the sides of the equation $\Rightarrow \sqrt{x^2} = \sqrt{2} \quad \Rightarrow \quad x = \pm\sqrt{2}$. Let us consider $x = \sqrt{2}$.

Can we represent $\sqrt{2}$ on number line ? What is the value of $\sqrt{2}$? To which numbers does $\sqrt{2}$ belong? Let us discuss.

Let us find the value of $\sqrt{2}$ by long division method.

	1.4142135
1	2. $\overline{00}$ $\overline{00}$ $\overline{00}$ $\overline{00}$ $\overline{00}$ $\overline{00}$ $\overline{00}$
	1
24	100
	96
281	400
	281
2824	11900
	11296
28282	60400
	56564
282841	383600
	282841
2828423	10075900
	8485269
28284265	159063100
	141421325
28284270	17641775

Step 1 : After 2, place decimal point.

Step 2 : After decimal point write 0's.

Step 3 : Group '0' in pairs and put a bar over them.

Step 4 : Then follow the method to find the square root of perfect square.

$\therefore \sqrt{2} = 1.4142135 \ldots$

If you go on finding the value of $\sqrt{2}$, you observe that $\sqrt{2}$ =1.4142135623731..... is neither terminating nor repeating decimal.

So far we have observed that the decimal number is either terminating or non-terminating repeating decimal, which can be expressed in $\frac{p}{q}$ form. These are known as rational numbers.

But decimal number for $\sqrt{2}$ is non-terminating and non-recurring decimal. Can you represent this $\sqrt{2} = 1.414213$ using bar? No we can't. These type of numbers are called irrational numbers and they can't be represented in $\frac{p}{q}$ form where p, q are integers and $q \neq 0$. That is $\sqrt{2} \neq \frac{p}{q}$ (for any integers p and q, $q \neq 0$).

Similarly $\sqrt{3} = 1.7320508075689.....$

$\sqrt{5} = 2.2360679774998.....$

These are non-terminating and non-recurring decimals. These are known as irrational numbers and are denoted by 'S' or 'Q^1'.

Examples of irrational numbers

(1) 2.1356217528..., (2) $\sqrt{2}$, $\sqrt{3}$, π, etc.

In 5th Century BC the Pythagoreans in Greece, the followers of the famous mathematician and philosopher Pythagoras, were the first to discover the numbers which were not rationals. These numbers are called irrational numbers. The Pythagoreans proved that $\sqrt{2}$ is irrational number. Later Theodorus of Cyrene showed that $\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}$ and $\sqrt{17}$ are also irrational numbers. There is a reference of irrationals in calculation of square roots in Sulba Sutra (800 BC).

Observe the following table

$\sqrt{1}$	=	1
$\sqrt{2}$	=	1.414213.....
$\sqrt{3}$	=	1.7320508.....
$\sqrt{4}$	=	2
$\sqrt{5}$	=	2.2360679.....
$\sqrt{6}$	=	
$\sqrt{7}$	=	
$\sqrt{8}$	=	
$\sqrt{9}$	=	3

If 'n' is a natural number other than a perfect square then $\sqrt{n}$ is an irrational number

Now, can you classify the numbers in the table as rational and irrational numbers?

$\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$ - are rational numbers.

$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$ - are irrational numbers.

THINK DISCUSS AND WRITE

Kruthi said $\sqrt{2}$ can be written as $\frac{\sqrt{2}}{1}$ which is in $\frac{p}{q}$ form. So $\sqrt{2}$ is a rational number. Do you agree with her argument ?

Know About π

π is defined as the ratio of the circumference (C) of a circle to its diameter (d). i.e. $\pi = \frac{c}{d}$

As π is in the form of $\frac{p}{q}$, this seems to contradict the fact that π is irrational. The circumference (C) and the diameter (d) of a circle are incommensurable. i.e. there does not exist a common unit to measure that allows us to measure the both numerator and denominator. If you measure accurately then atleast either **C** or **d** is irrational. So π is regarded as irrational.

The Greek genius **Archimedes** was the first to compute the value of π. He showed the value of π lie between 3.140845 and 3.142857. (i.e., $3.140845 < \pi < 3.142857$) **Aryabhatta** (476-550 AD), the great Indian mathematician and astronomer, found the value of π correctly upto four decimal places 3.1416. Using high speed computers and advanced algorithms, π has been computed to over 1.24 trillion decimal places .

π =3.14159265358979323846264338327950 ….. The decimal expansion of π is non-terminating non-recurring. So π is an irrational number. Note that, we often take $\frac{22}{7}$ as an approximate value of π, but $\pi \neq \frac{22}{7}$.

We celebrate March 14th as π day since it is 3.14 (as $\pi = 3.14159$ ....). What a coincidence, **Albert Einstein** was born on March 14th, 1879!

TRY THIS

Find the value of $\sqrt{3}$ upto six decimals. by long division method.

1.2.1 Representing irrational numbers on Number line

We have learnt that there exist a rational number between any two rational numbers. Therefore, when two rational numbers are represented by points on number line, we can use a point to represent a rational number between them. So there are infinitely many points representing rational numbers. It seems that the number line is consisting of points which represent rational numbers only. Is it true? Can't you represent $\sqrt{2}$ on number line? Let us discuss and locate irrational numbers such as $\sqrt{2}$, $\sqrt{3}$ on the number line.

Example-7. Locate $\sqrt{2}$ on number line

Solution : At O draw a unit square OABC on number line with each side 1 unit in length.

By Pythagoras theorem OB = $\sqrt{1^2+1^2} = \sqrt{2}$

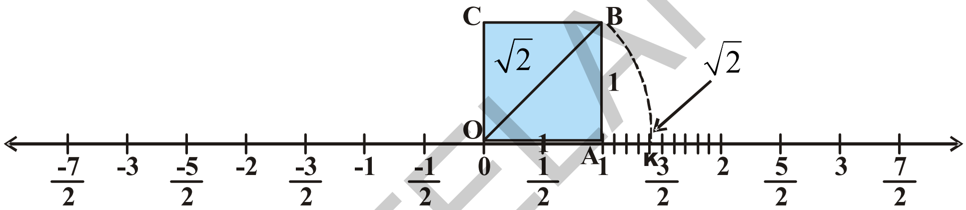


Fig. (i)

We have seen that OB = $\sqrt{2}$. Using a compass with centre O and radius OB, draw an arc on the right side to O intersecting the number line at the point K. Now K corresponds to $\sqrt{2}$ on the number line.

Example-8. Locate $\sqrt{3}$ on the number line.

Solution : Let us return to fig. (i)

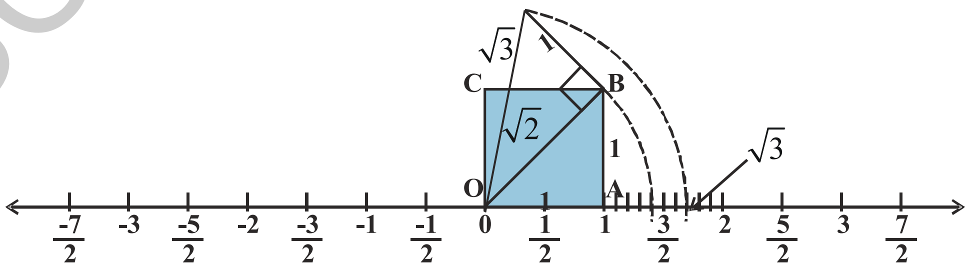


Fig. (ii)

Now construct BD of 1 unit length perpendicular to OB as in Fig. (ii). Join OD

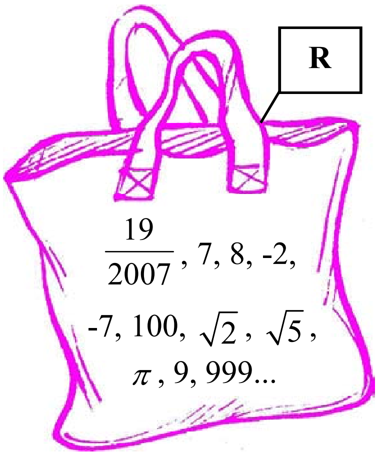
By Pythagoras theorem, OD = $\sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{2+1} = \sqrt{3}$

Using a compass, with centre O and radius OD, draw an arc which intersects the number line at the point L right side to 0. Then 'L' corresponds to $\sqrt{3}$. From this we can conclude that many points on the number line can be represented by irrational numbers also. In the same way, we can locate $\sqrt{n}$ for any positive integers n, after $\sqrt{n-1}$ has been located.

Try These

Locate $\sqrt{5}$ and $-\sqrt{5}$ on number line. [**Hint** : $5^2 = (2)^2 + (1)^2$]

1.3 Real Numbers



All rational numbers can be written in the form of $\frac{p}{q}$, where p and q are integers and q $\neq$ 0. There are also other numbers that cannot be written in the form $\frac{p}{q}$, where p and q are integers and are called irrational numbers. If we represent all rational numbers and all irrational numbers and put these on the number line, would there be any point on the number line that is not covered? The answer is no! All pts on the naumber line represents either rational and irrational numbers. This combination of rational number and irrational numbers are called Real Numbers, denoted by R. We can say that every real number is represented by a unique point on the number line. Also, every point on the number line represents a unique real number. So we call this as the real number line.

Here are some examples of Real numbers

$-5.6, \sqrt{21}, -2, 0, 1, \frac{1}{5}, \frac{22}{7}, \pi, \sqrt{2}, \sqrt{7}, \sqrt{9}$, 12.5, 12.5123..... etc. You may find that both rational and irrationals are included in this collection.

Example-9 : Find any two irrational numbers between $\frac{1}{5}$ and $\frac{2}{7}$.

Solution : We know that $\frac{1}{5} = 0.20$

$$\frac{2}{7} = 0.\overline{285714}$$

To find two irrational numbers between $\frac{1}{5}$ and $\frac{2}{7}$, we need to look at the decimal form of the two numbers and then proceed. We can find infinitely many such irrational numbers.

Examples of two such irrational numbers are

0.201201120111..., 0.24114111411114…, 0.25231617181912..., 0.267812147512 …

Can you find four more irrational numbers between $\frac{1}{5}$ and $\frac{2}{7}$?

Example-10. Find an irrational number between 3 and 4.

Solution :

> If a and b are two positive rational numbers such that ab is not a perfect square of a rational number, then $\sqrt{ab}$ is an irrational number lying between *a* and *b*.

$\therefore$ An irrational number between 3 and 4 is $\sqrt{3\times4} = \sqrt{3}\times\sqrt{4}$

$$= \sqrt{3}\times 2 = 2\sqrt{3}$$

Example-11. Examine, whether the following numbers are rational or irrational :

(i) $\left(3+\sqrt{3}\right)+\left(3-\sqrt{3}\right)$ (ii) $\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)$

(iii) $\frac{10}{2\sqrt{5}}$ (iv) $\left(\sqrt{2}+2\right)^2$

Solution :

(i) $\left(3+\sqrt{3}\right)+\left(3-\sqrt{3}\right) = 3+\sqrt{3}+3-\sqrt{3}$

$= 6$, which is a rational number.

(ii) $\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)$

We know that $\left(a+b\right)\left(a-b\right) \equiv a^2 - b^2$ is an identity.

Thus $\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right) = 3^2 - \left(\sqrt{3}\right)^2 = 9 - 3 = 6$ which is a rational number.

(iii) $\dfrac{10}{2\sqrt{5}} = \dfrac{10 \div 2}{2\sqrt{5} \div 2} = \dfrac{5}{\sqrt{5}} = \dfrac{\sqrt{5} \times \sqrt{5}}{\sqrt{5}} = \sqrt{5}$, which is an irrational number.

(iv) $\left(\sqrt{2}+2\right)^2 = \left(\sqrt{2}\right)^2 + 2.\sqrt{2}\ .2 + 2^2 = 2 + 4\sqrt{2} + 4$

$= 6 + 4\sqrt{2}$, which is an irrational number.

Exercise - 1.2

1. Classify the following numbers as rational or irrational.

(i) $\sqrt{27}$ (ii) $\sqrt{441}$ (iii) $30.232342345\ldots$

(iv) $7.484848\ldots$ (v) 11.2132435465 (vi) $0.3030030003.....$

2. Give four examples for rational and irrational numbers?

3. Find an irrational number between $\frac{5}{7}$ and $\frac{7}{9}$. How many more there may be?

4. Find two irrational numbers between 0.7 and 0.77

5. Find the value of $\sqrt{5}$ upto 3 decimal places.

6. Find the value of $\sqrt{7}$ up to six decimal places by long division method.

7. Locate $\sqrt{10}$ on the number line.

8. Find atleast two irrational numbers between 2 and 3.

9. State whether the following statements are true or false. Justify your answers with an example.

(i) Every irrational number is a real number.

(ii) Every rational number is a real number.

(iii) Every real number need not be a rational number

(iv) $\sqrt{n}$ is not irrational if n is a perfect square.

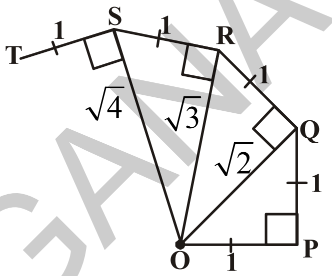
(v) $\sqrt{n}$ is irrational if n is not a perfect square.

(vi) All real numbers are irrational.

Activity

Constructing the 'Square root spiral'.

Take a large sheet of paper and construct the 'Square root spiral' in the following manner.



Step 1 : Start with point 'O' and draw a line segment $\overline{OP}$ of 1 unit length.

Step 2 : Draw a line segment $\overline{PQ}$ perpendicular to $\overline{OP}$ of unit length (where OP = PQ = 1) (see Fig)

Step 3 : Join O, Q. (OQ = $\sqrt{2}$)

Step 4 : Draw a line segment $\overline{QR}$ of unit length perpendicular to $\overline{OQ}$.

Step 5 : Join O, R. (OR = $\sqrt{3}$)

Step 6 : Draw a line segment RS of unit length perpendicular to $\overline{OR}$.

Step 7 : Continue in this manner for some more number of steps, you will create a beautiful spiral made of line segments $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, $\overline{ST}$, $\overline{TU}$ ... etc. Note that the line segments $\overline{OQ}$, $\overline{OR}$, $\overline{OS}$, $\overline{OT}$, $\overline{OU}$ ... etc. denote the lengths $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}$ respectively.

1.4 Representing Real numbers on the Number line through Successive magnification

In the previous section, we have seen that any real number has a decimal expansion.

Now first let us see how to represent terminating decimal on the number line.

Suppose we want to locate 2.776 on the number line. We know that this is a terminating decimal and this lies between 2 and 3.

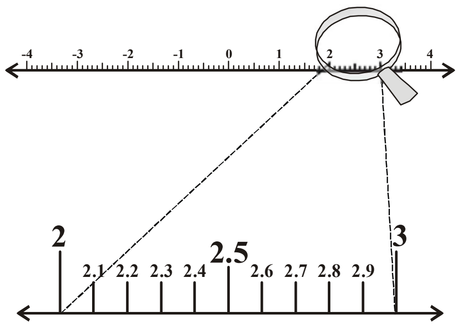


Fig.(i)

So, let us look closely at the portion of the number line between 2 and 3. Suppose we divide this into 10 equal parts as in Fig.(i). Then the markings will be like 2.1, 2.2, 2.3 and so on. To have a clear view, let us assume that we have a magnifying glass in our hand and look at the portion between 2 and 3. It will look like what you see in figure (i).

Now, 2.776 lies between 2.7 and 2.8. So, let us focus on the portion between 2.7 and 2.8 (See Fig. (ii). We imagine that this portion has been divided into ten equal parts. The first mark will represent 2.71, the second is 2.72, and so on. To see this clearly, use magnifying glass as shown in Fig(ii).

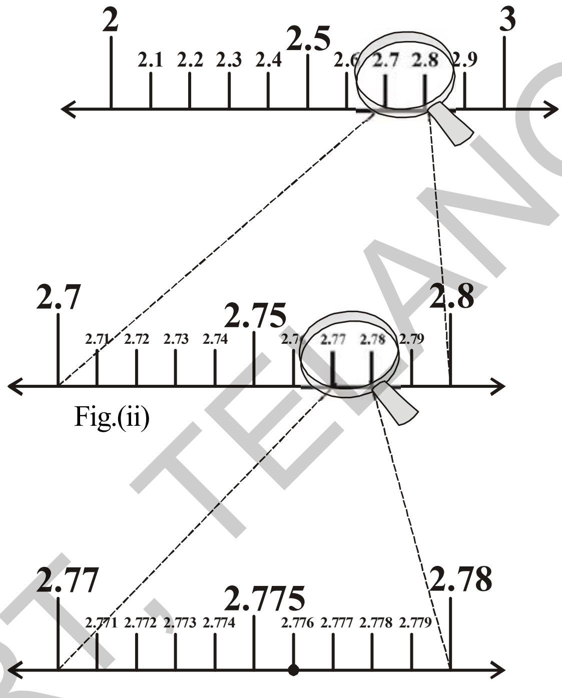


Fig.(iii)

Again 2.776 lies between 2.77 and 2.78. So, let us focus on this portion of the number line see Fig. (iii) and imagine that it has been divided again into ten equal parts. We magnify it to see it better, as in Fig.(iii).

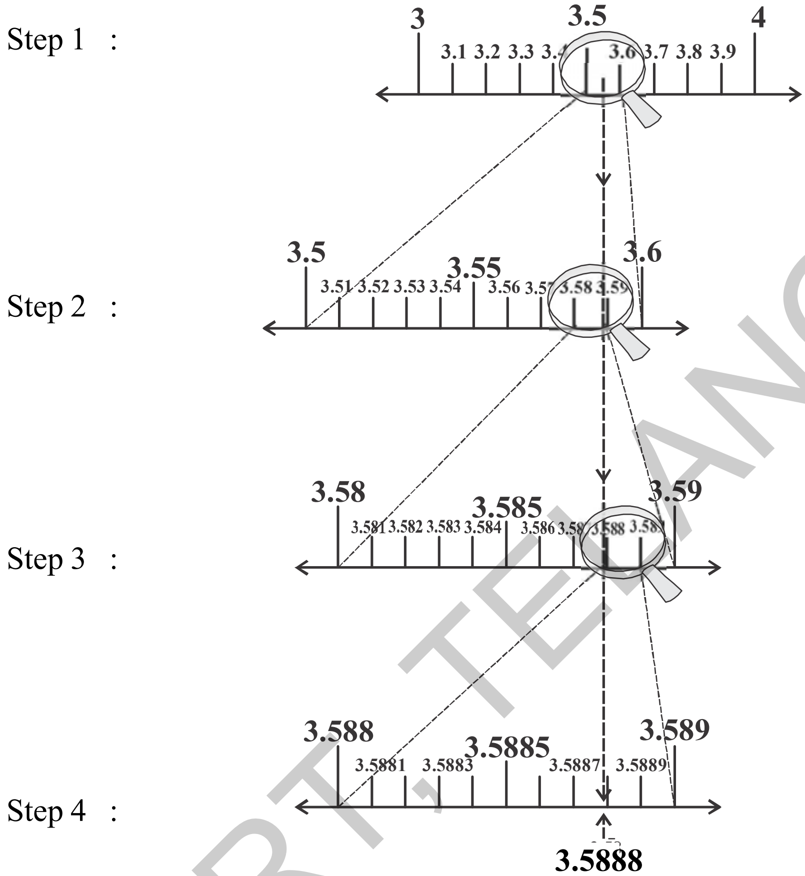
The first mark represents 2.771, second mark 2.772 and so on, 2.776 is the 6^{th} mark in these subdivisions.

We call this process of visualization of presentation of numbers on the number line through a magnifying glass, as the process of successive magnification.

Now let us try and visualize the position of a real number with a non-terminating recurring decimal expansion on the number line by the process of successive magnification with the following example.

Example-12. Visualise the representation of $3.5\overline{8}$ on the number line through successive magnification upto 4 decimal places.

Solution: Once again we proceed with the method of successive magnification to represent 3.5888 on number line.



Exercise - 1.3

1. Visualise 2.874 on the number line, using successive magnification.
2. Visualilse $5.\overline{28}$ on the number line, upto 3 decimal places.

1.5 Operations on Real Numbers

We have learnt, in previous class, that rational numbers satisfy the commutative, associative and distributive laws under addition and multiplication. And also, we learnt that rational numbers are closed with respect to addition, subtraction, multiplication. Can you say irrational numbers are also closed under four fundamental operations?

Look at the following examples

$\left(\sqrt{3}\right)+\left(-\sqrt{3}\right)=0$. Here 0 is a rational number.

$\left(\sqrt{5}\right)-\left(\sqrt{5}\right)=0$. Here 0 is a rational number.

$\left(\sqrt{2}\right).\left(\sqrt{2}\right)=2$. Here 2 is a rational number.

$\frac{\sqrt{7}}{\sqrt{7}}=1$. Here 1 is a rational number.

What do you observe? The sum, difference, products and quotients of irrational numbers need not be irrational numbers.

So we can say irrational numbers are not closed with respect to addition, subtraction, multiplication and divisioin.

Suppose you want to add $3\sqrt{2}$ to $2\sqrt{2}$, the sum can be written as $5\sqrt{2}$ and if you subtract $2\sqrt{2}$ from $3\sqrt{2}$, you will get $\sqrt{2}$

Think, Discuss and Write

1. Hasith said "$5\sqrt{3}+2\sqrt{7}=7\sqrt{10}$". Do you agree with him?
2. How can you find the value of $5\sqrt{2}-\sqrt{8}$?

Let us see some problems on irrational numbers.

Example-13. Check whether (i) $5\sqrt{2}$ (ii) $\frac{5}{\sqrt{2}}$ (iii) $21+\sqrt{3}$ (iv) $\pi+3$ are irrational numbers or not?

Solution : We know $\sqrt{2}=1.414\ldots$, $\sqrt{3}=1.732\ldots$, $\pi=3.1415\ldots$

(i) $5\sqrt{2} = 5(1.414\ldots)=7.070\ldots$

(ii) $\frac{5}{\sqrt{2}} = \frac{5}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{5\sqrt{2}}{2}=\frac{7.070}{2}=3.535\ldots$ (from i)

(iii) $21+\sqrt{3} = 21+1.732\ldots = 22.732\ldots$

(iv) $\pi+3 = 3.1415\ldots+3=6.1415\ldots$

All these are non-terminating, non-recurring decimals.

Thus they are irrational numbers.

If q is rational and s is irrational then q + s, q - s, qs and $\frac{q}{s}$ are irrational numbers

Example-14. Subtract $5\sqrt{3}+7\sqrt{5}$ from $3\sqrt{5}-7\sqrt{3}$

Solution : $(3\sqrt{5}-7\sqrt{3})-(5\sqrt{3}+7\sqrt{5})$

$= 3\sqrt{5}-7\sqrt{3}-5\sqrt{3}-7\sqrt{5}$

$= -4\sqrt{5}-12\sqrt{3}$

$= -(4\sqrt{5}+12\sqrt{3})$

Example-15. Multiply $6\sqrt{3}$ with $13\sqrt{3}$

Solution : $6\sqrt{3} \times 13\sqrt{3} = 6 \times 13 \times \sqrt{3} \times \sqrt{3} = 78 \times 3 = 234$

Properties relating to square roots are given belwo, which are useful in various ways.

Let a and b be non-negative real numbers. Then

(i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(ii) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$; if $b \neq 0$

(iii) $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = a - b$

(iv) $\left(a+\sqrt{b}\right)\left(a-\sqrt{b}\right) = a^2 - b$

(v) $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{c}+\sqrt{d}\right) = \sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}$

(vi) $\left(\sqrt{a}+\sqrt{b}\right)^2 = a + 2\sqrt{ab} + b$

(vii) $\sqrt{a+b+2\sqrt{ab}} = \sqrt{a}+\sqrt{b}$

Let us look at some particular cases of these properties.

Example-16. Simplify the following expressions:

(i) $\left(3+\sqrt{3}\right)\left(2+\sqrt{2}\right)$ (ii) $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)$

(iii) $\left(\sqrt{5}+\sqrt{2}\right)^2$ (iv) $\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right)$

Solution :

(i) $\left(3+\sqrt{3}\right)\left(2+\sqrt{2}\right) = 6+3\sqrt{2}+2\sqrt{3}+\sqrt{6}$

(ii) $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right) = 2^2-\left(\sqrt{3}\right)^2 = 4 - 3 = 1$

(iii) $\left(\sqrt{5}+\sqrt{2}\right)^2 = \left(\sqrt{5}\right)^2+2\sqrt{5}\sqrt{2}+\left(\sqrt{2}\right)^2 = 5+2\sqrt{10}+2 = 7+2\sqrt{10}$

(iv) $\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right) = \left(\sqrt{5}\right)^2-\left(\sqrt{2}\right)^2 = 5-2 = 3$

Example-17. Find the square root of $5+2\sqrt{6}$

Solution : $\sqrt{5+2\sqrt{6}}$

$= \sqrt{3+2+2\cdot\sqrt{3}\cdot\sqrt{2}}$ $\quad \because \sqrt{a+b+2\sqrt{ab}} = \sqrt{a}+\sqrt{b}$

$= \sqrt{3}+\sqrt{2}$

1.5.1 Rationalising the Denominator

Can we locate $\frac{1}{\sqrt{2}}$ on the number line ?

What is the value of $\frac{1}{\sqrt{2}}$?

How do we find the value? As $\sqrt{2} = 1.4142135.....$ which is neither terminating nor repeating. Can you divide 1 with this?

It does not seem to be easy to find $\frac{1}{\sqrt{2}}$.

Let us try to change the denominator into a rational form. To rationalise the denominator of $\frac{1}{\sqrt{2}}$, multiply the numerator and the denominator of $\frac{1}{\sqrt{2}}$ by $\sqrt{2}$, we get

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$. Yes, it is half of $\sqrt{2}$.

Now can we plot $\frac{\sqrt{2}}{2}$ on the number line ? It lies between 0(zero) and $\sqrt{2}$.

Observe that $\sqrt{2} \times \sqrt{2} = 2$. Thus we say $\sqrt{2}$ is the rationalising factor (R.F) of $\sqrt{2}$

Similarly $\sqrt{2} \times \sqrt{8} = \sqrt{16} = 4$. Then $\sqrt{2}$ and $\sqrt{8}$ are rationalising factors of each other $\sqrt{2} \times \sqrt{18} = \sqrt{36} = 6$, etc. Therefore $\sqrt{2}$ is rationalising factor of $\sqrt{18}$. From above examples $\sqrt{2}$ is a rationalising factor of $\sqrt{2}$, $\sqrt{8}$, $\sqrt{18}$, $\sqrt{32}$, ... etc. Among these $\sqrt{2}$ is the simplest rationalising factor of $\sqrt{2}$.

Note that if the product of two irrational numbers is a rational number then each of the two is the rationalising factor (R.F) of the other. Also notice that the R.F. of a given irrational number is not unique. It is convenient to use the simplest of all R.F.s of given irrational number.

Do This

Find rationalising factors of the denominators of (i) $\frac{1}{2\sqrt{3}}$ (ii) $\frac{3}{\sqrt{5}}$ (iii) $\frac{1}{\sqrt{8}}$.

Example-18. Rationalise the denominator of $\frac{1}{4+\sqrt{5}}$

Solution : We know that $\left(a+\sqrt{b}\right)\left(a-\sqrt{b}\right)=a^2-b$

Multiplying the numerator and denominator of $\frac{1}{4+\sqrt{5}}$ by $4-\sqrt{5}$

$$\frac{1}{4+\sqrt{5}}\times\frac{4-\sqrt{5}}{4-\sqrt{5}}=\frac{4-\sqrt{5}}{4^2-\left(\sqrt{5}\right)^2}=\frac{4-\sqrt{5}}{16-5}=\frac{4-\sqrt{5}}{11}$$

Example-19. If $x=7+4\sqrt{3}$ then find the value of $x+\frac{1}{x}$

Solution : Given $x=7+4\sqrt{3}$

Now $$\frac{1}{x}=\frac{1}{7+4\sqrt{3}}\times\frac{7-4\sqrt{3}}{7-4\sqrt{3}}=\frac{7-4\sqrt{3}}{7^2-\left(4\sqrt{3}\right)^2}=\frac{7-4\sqrt{3}}{49-16\times 3}$$

$$=\frac{7-4\sqrt{3}}{49-48}=7-4\sqrt{3}$$

$$\therefore x+\frac{1}{x}=7+4\sqrt{3}+7-4\sqrt{3}=14$$

Example-20. Simplify $\frac{1}{7+4\sqrt{3}}+\frac{1}{2+\sqrt{5}}$

Solution : The rationalising factor of $7+4\sqrt{3}$ is $7-4\sqrt{3}$ and the rationalising factor of $2+\sqrt{5}$ is $2-\sqrt{5}$.

$$=\frac{1}{7+4\sqrt{3}}+\frac{1}{2+\sqrt{5}}$$

$$=\frac{1}{7+4\sqrt{3}}\times\frac{7-4\sqrt{3}}{7-4\sqrt{3}}+\frac{1}{2+\sqrt{5}}\times\frac{2-\sqrt{5}}{2-\sqrt{5}}$$

$$=\frac{7-4\sqrt{3}}{7^2-(4\sqrt{3})^2}+\frac{2-\sqrt{5}}{2^2-(\sqrt{5})^2}$$

$$=\frac{7-4\sqrt{3}}{49-48}+\frac{2-\sqrt{5}}{(4-5)}$$

$$=\frac{7-4\sqrt{3}}{1}+\frac{2-\sqrt{5}}{(-1)}$$

$$=7-4\sqrt{3}-2+\sqrt{5}=5-4\sqrt{3}+\sqrt{5}$$

1.5.2 Law of Exponents for real numbers

Let us recall the laws of exponents.

i) $a^m . a^n = a^{m+n}$ ii) $(a^m)^n = a^{mn}$ iii) $\frac{a^m}{a^n} = \begin{cases} a^{m-n} & \text{if } m > n \\ 1 & \text{if } m = n \\ \frac{1}{a^{n-m}} & \text{if } m < n \end{cases}$

iv) $a^m b^m = (ab)^m$ v) $\frac{1}{a^n} = a^{-n}$ vi) $a^0 = 1 \quad (a \neq 0)$

Here a, b 'm' and 'n' are integers and a, b $\neq$ 0, *a, b* are called the base and m, n are the exponents.

For example

i) $7^3.7^{-3} = 7^{3+(-3)} = 7^0 = 1$ ii) $(2^3)^{-7} = 2^{-21} = \frac{1}{2^{21}}$

iii) $\frac{23^{-7}}{23^4} = 23^{-7-4} = 23^{-11}$ iv) $(7)^{-13}.(3)^{-13} = (7\times3)^{-13} = (21)^{-13}$

Let us compute the following.

i) $2^{\frac{2}{3}}.2^{\frac{1}{3}}$ ii) $\left(5^{\frac{1}{7}}\right)^4$ iii) $\frac{3^{\frac{1}{5}}}{3^{\frac{1}{3}}}$ iv) $7^{\frac{1}{17}}.11^{\frac{1}{17}}$

How do we go about it? The exponents and bases in the above examples are rational numbers. We cant find there using previous discussed lawas of exponents.Thus there is a need to extend the laws of exponents to bases of positive real numbers and to the exponents as rational numbers. Before we state these laws, we need first to understand what is nth root of a real number.

We know if $3^2 = 9$ then $\sqrt{9} = 3$ (square root of 9 is 3)

i.e., $\sqrt[2]{9} = 3$

If $5^2 = 25$ then $\sqrt{25} = 5$ i.e., $\sqrt[2]{25} = 5$ moreover $\sqrt[2]{25} = (25)^{\frac{1}{2}} = \left(5^2\right)^{\frac{1}{2}} = 5^{2\times\frac{1}{2}} = 5$

Observe the following

If $2^3 = 8$ then $\sqrt[3]{8} = 2$ (cube root of 8 is 2); $\sqrt[3]{8} = 8^{\frac{1}{3}} = \left(2^3\right)^{\frac{1}{3}} = 2$

$2^4 = 16$ then $\sqrt[4]{16} = 2$ (4th root of 16 is 2); $\sqrt[4]{16} = (16)^{\frac{1}{4}} = \left(2^4\right)^{\frac{1}{4}} = 2$

$2^5 = 32$ then $\sqrt[5]{32} = 2$ (5^{th} root of 32 is 2); $\sqrt[5]{32} = (32)^{\frac{1}{5}} = \left(2^5\right)^{\frac{1}{5}} = 2$

$2^6 = 64$ then $\sqrt[6]{64} = 2$ (6^{th} root of 64 is 2); $\sqrt[6]{64} = (64)^{\frac{1}{6}} = \left(2^6\right)^{\frac{1}{6}} = 2$

....

Similarly if $\mathbf{a^n = b}$ then $\sqrt[n]{\mathbf{b}} = \mathbf{a}$ (n^{th} root of b is a); $\sqrt[n]{b} = (b)^{\frac{1}{n}} = \left(a^n\right)^{\frac{1}{n}} = a$

Let $\mathbf{a} > 0$ be a real number and '**n**' be a positive integer.

If $\mathbf{b^n = a}$, for some positive real number '**b**', then **b** is called nth root of '**a**' and we write $\sqrt[n]{\mathbf{a}} = \mathbf{b}$. In the earlier discussion laws of exponents were defined for integers. Let us extend the laws of exponents to the bases of positive real numbers and rational exponents.

Let a > 0 be a real number and p and q be rational numbers then, we have

i) $a^p . a^q = a^{p+q}$ ii) $(a^p)^q = a^{pq}$

iii) $\dfrac{a^p}{a^q} = a^{p-q}$ iv) $a^p . b^p = (ab)^p$ v) $\sqrt[n]{a} = a^{\frac{1}{n}}$

Now we can use these laws to answer the questions asked earlier.

Example-21. Simplify

i) $2^{\frac{2}{3}} . 2^{\frac{1}{3}}$ ii) $\left(5^{\frac{1}{7}}\right)^4$ iii) $\dfrac{3^{\frac{1}{5}}}{3^{\frac{1}{3}}}$ iv) $7^{\frac{1}{17}} . 11^{\frac{1}{17}}$

Solution : i) $2^{\frac{2}{3}} . 2^{\frac{1}{3}} = 2^{\left(\frac{2}{3}+\frac{1}{3}\right)} = 2^{\frac{3}{3}} = 2^1 = 2$

ii) $\left(5^{\frac{1}{7}}\right)^4 = 5^{\frac{4}{7}}$

iii) $\dfrac{3^{\frac{1}{5}}}{3^{\frac{1}{3}}} = 3^{\left(\frac{1}{5}-\frac{1}{3}\right)} = 3^{\frac{3-5}{15}} = 3^{\frac{-2}{15}} = \dfrac{1}{3^{2/15}}$

iv) $7^{\frac{1}{17}} . 11^{\frac{1}{17}} = (7 \times 11)^{\frac{1}{17}} = 77^{\frac{1}{17}}$

Do This

Simplify:

i. $(16)^{\frac{1}{2}}$ ii. $(128)^{\frac{1}{7}}$

iii. $(343)^{\frac{1}{5}}$

Surd

If '**n**' is a positive integer greater than 1 and '**a**' is a positive rational number but not $\mathbf{n}^{th}$ power of any rational number then $\sqrt[n]{\mathbf{a}}$ (or) $\mathbf{a}^{1/n}$ is called a surd of $\mathbf{n}^{th}$ order. In general we say the positive $\mathbf{n}^{th}$ root of **a** is called a surd or a radical. Here a is called radicand, $\sqrt[n]{\ }$ is called radical sign and **n** is called the degree of radical.

Forms of Surd

Exponential form $a^{\frac{1}{n}}$

Radical form $\sqrt[n]{a}$

Here are some examples for surds.

$\sqrt{2}, \sqrt{3}, \sqrt[3]{9},$ ..... etc

Consider the real number $\sqrt{7}$. It may also be written as $7^{\frac{1}{2}}$. Since 7 is not a square of any rational number, $\sqrt{7}$ is a surd.

Consider the real number $\sqrt[3]{8}$. Since 8 is a cube of a rational number 2, So, $\sqrt[3]{8}$ is not a surd.

Consider the real number $\sqrt{\sqrt{2}}$. It may be written as $\left(2^{\frac{1}{2}}\right)^{\frac{1}{2}} = 2^{\frac{1}{4}} = \sqrt[4]{2}$. 2 can not be written as fourth root of any rational number. So it is a surd of 4th degree.

Do This

1. Write the following surds in exponential form

   i. $\sqrt{2}$ ii. $\sqrt[3]{9}$ iii. $\sqrt[5]{20}$ iv. $\sqrt[17]{19}$

2. Write the surds in radical form:

   i. $5^{\frac{1}{7}}$ ii. $17^{\frac{1}{6}}$ iii. $5^{\frac{2}{5}}$ iv. $142^{\frac{1}{2}}$

Exercise - 1.4

1. Simplify the following expressions.

i) $\left(5+\sqrt{7}\right)\left(2+\sqrt{5}\right)$ ii) $\left(5+\sqrt{5}\right)\left(5-\sqrt{5}\right)$

iii) $\left(\sqrt{3}+\sqrt{7}\right)^2$ iv) $\left(\sqrt{11}-\sqrt{7}\right)\left(\sqrt{11}+\sqrt{7}\right)$

2. Classify the following numbers as rational or irrational.

i) $5-\sqrt{3}$ ii) $\sqrt{3}+\sqrt{2}$ iii) $\left(\sqrt{2}-2\right)^2$

iv) $\frac{2\sqrt{7}}{7\sqrt{7}}$ v) 2π vi) $\frac{1}{\sqrt{3}}$ vii) $(2+\sqrt{2})(2-\sqrt{2})$

3. In the following equations, find whether variables x, y, z etc. represent rational or irrational numbers

i) $x^2 = 7$ ii) $y^2 = 16$ iii) $z^2 = 0.02$

iv) $u^2 = \frac{17}{4}$ v) $w^2 = 27$ vi) $t^4 = 256$

4. Every surd is an irrational, but every irrational need not be a surd. Justify your answer.

5. Rationalise the denominators of the following:

i) $\frac{1}{3+\sqrt{2}}$ ii) $\frac{1}{\sqrt{7}-\sqrt{6}}$ iii) $\frac{1}{\sqrt{7}}$ iv) $\frac{\sqrt{6}}{\sqrt{3}-\sqrt{2}}$

6. Simplify each of the following by rationalising the denominator:

i) $\frac{6-4\sqrt{2}}{6+4\sqrt{2}}$ ii) $\frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$ iii) $\frac{1}{3\sqrt{2}-2\sqrt{3}}$ iv) $\frac{3\sqrt{5}-\sqrt{7}}{3\sqrt{3}+\sqrt{2}}$

7. Find the value of $\frac{\sqrt{10}-\sqrt{5}}{2\sqrt{2}}$ upto three decimal places. (take $\sqrt{2} = 1.414$ and $\sqrt{5} = 2.236$)

8. Find the Values.

i) $64^{\frac{1}{6}}$ ii) $32^{\frac{1}{5}}$ iii) $625^{\frac{1}{4}}$

iv) $16^{\frac{3}{2}}$ v) $243^{\frac{2}{5}}$ vi) $(46656)^{\frac{-1}{6}}$

9. Simplify: $\sqrt[4]{81} - 8\sqrt[3]{343} + 15\sqrt[5]{32} + \sqrt{225}$

10. If 'a' and 'b' are rational numbers, find the value of a and b in each of the following equations.

i) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} = a + b\sqrt{6}$ ii) $\frac{\sqrt{5}+\sqrt{3}}{2\sqrt{5}-3\sqrt{3}} = a - b\sqrt{15}$

11. Find the square root of $11+2\sqrt{30}$

What we have discussed ?

In this chapter we have discussed the following points:

1. A number which can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$ is called a rational number.
2. A number which cannot be written in the form $\frac{p}{q}$, for any integers p, q and $q \neq 0$ is called an irrational number.
3. The decimal expansion of a rational number is either terminating or non-terminating recurring.
4. The decimal expansion of an irrational number is non-terminating and non-recurring.
5. The collection of all rational and irrational numbers is called Real numbers.
6. There is a unique real number corresponding to every point on the number line. Also corresponding to each real number, there is a unique point on the number line.
7. If q is rational and s is irrational, then q+s, q-s, qs and $\frac{q}{s}$ are irrational numbers.
8. If n is a natural number other than a perfect square, then $\sqrt{n}$ is an irrational number.
9. The following identities hold for positive real numbers a and b

   i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$ ii) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ $(b \neq 0)$

   iii) $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = a - b$ iv) $\left(a+\sqrt{b}\right)\left(a-\sqrt{b}\right) = a^2 - b$

   v) $\left(\sqrt{a}+\sqrt{b}\right)^2 = a + 2\sqrt{ab} + b$ vi) $\sqrt{a+b+2\sqrt{ab}} = \sqrt{a}+\sqrt{b}$

10. To rationalise the denominator of $\frac{1}{\sqrt{a}+b}$, we multiply this by $\frac{\sqrt{a}-b}{\sqrt{a}-b}$, where a, b are integers.
11. Let $a > 0$, $b > 0$ be a real number and p and q be rational numbers. Then

    i) $a^p . a^q = a^{p+q}$ ii) $(a^p)^q = a^{pq}$ iii) $\frac{a^p}{a^q} = a^{p-q}$

    iv) $a^p . b^p = (ab)^p$

12. If 'n' is a positive integer > 1 and 'a' is a positive rational number but not n^{th} power of any rational number then $\sqrt[n]{a}$ or $a^{\frac{1}{n}}$ is called a surd of nth order.

Chapter

2 Polynomials and Factorisation

Z5H3C4

2.1 Introduction

There are six rows and each row has six plants in a garden. How many plants are there in total? If there are 'x' plants, planted in 'x' rows then how many plants will be there in the garden? Obviously it is x^2.

The cost of onions is ₹10 per kg. Inder purchased p kg., Raju purchased q kg. and Hanif purchased r kg. How much each would have paid? The payments would be ₹10p, ₹10q and ₹10r respectively. *All such examples show the use of algebraic expression.*

We also use algebraic expressions such as 's^2' to find area of a square, 'lb' for area of a rectangle and 'lbh' for volume of a cuboid. What are the other algebraic expressions that we use?

Algebraic expressions such as $3xy$, x^2+2x, $x^3 - x^2 + 4x + 3$, πr^2, $ax + b$ etc. are called polynomials. Note that, all algebraic expressions we have considered so far only have non-negative integers as exponents of the variables.

Can you find the polynomials among the given algebraic expressions:

$$x^2, \quad x^{\frac{1}{2}} + 3, \quad 2x^2 - \frac{3}{x} + 5; \quad x^2 + xy + y^2$$

From the above $x^{\frac{1}{2}} + 3$ and $2x^2 - \frac{3}{x} + 5$ are not polynomials. Because the first term $x^{\frac{1}{2}}$ is a term with an exponent that is not a non-negative integer (i.e. $\frac{1}{2}$) and also $2x^2 - \frac{3}{x} + 5$ is not a polynomial because it can be written as $2x^2 - 3x^{-1} + 5$. Here the second term ($3x^{-1}$) has a negative exponent. (i.e., -1). An algebraic expression in which the variables involved have only non-negative integral powers is called a polynomial.

Think, Discuss and Write

Which of the following expressions are polynomials ? Which are not ? Give reasons.

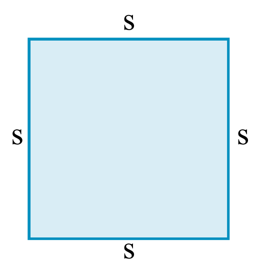
(i) $4x^2 + 5x - 2$ (ii) $y^2 - 8$ (iii) 5 (iv) $2x^2 + \frac{3}{x} - 5$

(v) $\sqrt{3}x^2 + 5y$ (vi) $\frac{1}{x} + 1 \; (x \neq 0)$ (vii) $\sqrt{x}$ (viii) $3\,xyz$

We shall start our study with polynomials in various forms. In this chapter we will also learn factorisation of polynomials using Remainder Theorem and Factor Theorem and their use in the factorisation of polynomials.

2.2 Polynomials in one Variable

Let us begin by recalling that a variable is denoted by a symbol that can take any real value. We use the letters x, y, z etc. to denote variables.



We have algebraic expressions such as $2x, 3x, -x, \frac{3}{4}x$ .... all in one variable x. These expressions are of the form (a constant) × (some power of variable). Now, suppose we want to find the perimeter of a square we use the formula P = 4s.

Here '4' is a constant and 's' is a variable, representing the side of a square. The side could vary for different squares.

Observe the following table:

Side of square	Perimeter
(s)	(4s)
4 cm	P = 4 × 4 = 16 cm
5 cm	P = 4 × 5 = 20 cm
10 cm	P = 4 × 10 = 40 cm

Here the value of the constant i.e. '4' remains the same throughout this situation. That is, the value of the constant does not change in a given problem, but the value of the variable (s) keeps changing.

Suppose we want to write an expression which is of the form '(a constant) × (a variable)' and we do not know, what the constant is, then we write the constants as a, b, c ... etc. So these expressions in general will be ax, by, cz, ... etc.

Here a, b, c ... are arbitrary constants. You are also familiar with other algebraic expressions like x^2, $x^2 + 2x + 1$, $x^3 + 3x^2 - 4x + 5$. All these expressions are polynomials in one variable.

Do This

- Write two polynomials with variable 'x'
- Write three polynomials with variable 'y'
- Is the polynomial $2x^2 + 3xy + 5y^2$ in one variable ?
- Write the formulae of area and volume of different solid shapes. Find out the variables and constants in them.

2.3 Degree of the polynomial

Each term of the polynomial consists of the product of a constant, (called the **coefficient** of the term) and a finite number of variables raised to non-negative integral powers. Degree of a term is the sum of the exponent of its variable factors. And the degree of a polynomial is the largest degree of its variable terms.

Lets find the terms, their coefficients and the degree of polynomials:

(i) $3x^2 + 7x + 5$ (ii) $3x^2y^2 + 4xy + 7$

In the polynomial $3x^2 + 7x + 5$, each of the expressions $3x^2$, $7x$ and 5 are terms. Each term of the polynomial has a coefficient, so in $3x^2 + 7x + 5$, the coefficient of x^2 is 3, the coefficient of x is 7 and 5 is the coefficient of x^0 (Remember x^0=1)

You know that the degree of a polynomial is the highest degree of its variable term.

As the term $3x^2$ has the highest degree among all the other terms in that expression, Thus the degree of $3x^2 + 7x + 5$ is '2'.

Now can you find coefficient and degree of polynomial $3x^2y^3 + 4xy + 7$.

The coefficient of x^2y^3 is 3, xy is 4 and x^0y^0 is 7. The sum of the exponents of the variables in term $3x^2y^3$ is $2 + 3 = 5$ which is greater than that of the other terms. So the degree of polynomial $3x^2y^3 + 4xy + 7$ is 5.

Now think what is the degree of a constant? As the constant contains no variable, it can be written as product of x^0. For example, degree of 5 is zero as it can be written as $5x^0$.

Now that you have seen what a polynomial of degree 1, degree 2, or degree 3 looks like, can you write down a polynomial in one variable of degree n for any natural number n? A polynomial in one variable x of degree n is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$.

In particular, if $a_0 = a_1 = a_2 = a_3 = \ldots = a_n = 0$ (i.e. all the coefficients are zero), we get the **zero polynomial**, which is denoted by 'O'.

Can you say the degree of zero? It is not defined as we can't write it as a product of a variable raised to any power.

Do this

1. Write the degree of each of the following polynomials
   (i) $7x^3 + 5x^2 + 2x - 6$ (ii) $7 - x + 3x^2$
   (iii) $5p - \sqrt{3}$ (iv) 2 (v) $-5xy^2$
2. Write the coefficient of x^2 in each of the following
   (i) $15 - 3x + 2x^2$ (ii) $1 - x^2$ (iii) $\pi x^2 - 3x + 5$ (iv) $\sqrt{2}x^2 + 5x - 1$

Let us observe the following tables and fill the blanks.

(i) Types of polynomials according to degree :

Degree of a polynomial	Name of the polynomial	Example
Not defined	Zero polynomial	0
Zero	Constant polynomial	$-12; 5; \frac{3}{4}$ etc
1	....	$x - 12; -7x + 8; ax + b$ etc.
2	Quadratic polynomial	....
3	Cubic polynomial	$3x^3 - 2x^2 + 5x + 7$

Usually, a polynomial of degree 'n' is called n^{th} degree polynomial.

(ii) Types of polynomials according to number of terms:

Number of non - zero terms	Name of the polynomial	Example	Terms
1	Monomial	$-3x$	$-3x$
2	Binomial	$3x + 5$	$3x, 5$
3	Trinomial	$2x^2 + 5x + 1$	....
More than 3	Multinomial	....	$3x^3, 2x^2, -7x, 5$

Note **:** A polynomial may be a multinomial but every multinomial need not be a polynomial.

A linear polynomial with one variable may be a monomial or a binomial.

Eg : $3x$ or $2x - 5$

Think, Discuss and Write

How many terms a cubic (degree 3) polynomial with one variable can have? Give examples.

If the variable in a polynomial is x, we may denote the polynomial by $p(x)$, $q(x)$ or $r(x)$ etc. So for example, we may write some polynomials in one variables.

$p(x) = 3x^2 + 2x + 1$

$q(x) = x^3 - 5x^2 + x - 7$

$r(y) = y^4 - 1$

$t(z) = z^2 + 5z + 3$

A polynomial can have any finite number of terms.

Try This

1. Write a polynomial with 2 terms in variable x.
2. How can you write a polynomial with 15 terms in variable p ?

So far mostly we have discussed the polynomials in one variable only. We can also have polynomials in more than one variable. For example $x+y$, $x^2+2xy+y^2$, x^2-y^2 are polynomials in two variables x, y. Similarly $x^2+y^2+z^2$, $x^3+y^3+z^3$ are polynomials in three variables. You will study such polynomials later in detail.

Exercise - 2.1

1. Find the degree of each of the polynomials given below

(i) $x^5 - x^4 + 3$ (ii) $x^2 + x - 5$ (iii) 5

(iv) $3x^6 + 6y^3 - 7$ (v) $4 - y^2$ (vi) $5t - \sqrt{3}$

2. Which of the following expressions are polynomials in one variable and which are not ? Give reasons for your answer.

(i) $3x^2 - 2x + 5$ (ii) $x^2 + \sqrt{2}$ (iii) $p^2 - 3p + q$ (iv) $y + \frac{2}{y}$ $(y \neq 0)$

(v) $5\sqrt{x} + x\sqrt{5}$ $(x>0)$ (vi) $x^{100} + y^{100}$

3. Write the coefficient of x^3 in each of the following

(i) $x^3 + x + 1$ (ii) $2 - x^3 + x^2$ (iii) $\sqrt{2}x^3 + 5$ (iv) $2x^3 + 5$

(v) $\frac{\pi}{2}x^3 + x$ (vi) $-\frac{2}{3}x^3$ (vii) $2x^2 + 5$ (viii) 4

4. Classify the following as linear, quadratic and cubic polynomials

(i) $5x^2 + x - 7$ (ii) $x - x^3$ (iii) $x^2 + x + 4$ (iv) $x - 1$

(v) $3p$ (vi) πr^2

5. Write whether the following statements are True or False. Justify your answer

(i) A binomial has two terms

(ii) Every polynomial is a binomial

(iii) A binomial may have degree 3

(iv) Degree of zero polynomial is zero

(v) The degree of polynomial $x^2 + 2xy + y^2$ is 2

(vi) πr^2 is a monomial.

6. Give one example each of a monomial and trinomial of degree 10.

2.4 (a) Zeroes of a polynomial

- Consider the polynomial $p(x) = x^2 + 5x + 4$.

What is the value of $p(x)$ at $x = 1$?

For this we have to replace x by 1 every where in $p(x)$

By doing this $p(1) = (1)^2 + 5(1) + 4,$

we get $= 1 + 5 + 4 = 10$

So, we say that the value of $p(x)$ at $x = 1$ is 10

Similarly find $p(x)$ for $x = 0$ and $x = -1$

$$\begin{aligned} p(0) &= (0)^2 + 5(0) + 4 \\ &= 0 + 0 + 4 \\ &= 4 \end{aligned} \qquad \begin{aligned} p(-1) &= (-1)^2 + 5(-1) + 4 \\ &= 1 - 5 + 4 \\ &= 0 \end{aligned}$$

Can you find the value of $p(-4)$?

- Consider another polynomial

$$\begin{aligned} s(y) &= 4y^4 - 5y^3 - y^2 + 6 \\ s(1) &= 4(1)^4 - 5(1)^3 - (1)^2 + 6 \\ &= 4(1) - 5(1) - 1 + 6 \\ &= 4 - 5 - 1 + 6 \\ &= 10 - 6 \\ &= 4 \end{aligned}$$

Can you find $s(-1)$?

Do This

Find the value of each of the following polynomials for the indicated value of variables:

(i) $p(x) = 4x^2 - 3x + 7$ at $x = 1$

(ii) $q(y) = 2y^3 - 4y + \sqrt{11}$ at $y = 1$

(iii) $r(t) = 4t^4 + 3t^3 - t^2 + 6$ at $t = p$, ($t \in R$)

(iv) $s(z) = z^3 - 1$ at $z = 1$

(v) $p(x) = 3x^2 + 5x - 7$ at $x = 1$

(vi) $q(z) = 5z^3 - 4z + \sqrt{2}$ at $z = 2$

- Now consider the polynomial $r(t) = t - 1$

What is $r(1)$? It is $r(1) = 1 - 1 = 0$

As $r(1) = 0$, we say that 1 is a zero of the polynomial $r(t)$.

In general, we say that a **zero of a polynomial** $p(x)$ is the value of x, for which $p(x) = 0$.

This value is also called a root of the polynomial $p(x)=0$

What is the zero of the polynomial $f(x) = x + 1$?

You must have observed that the zero of the polynomial $x + 1$ is obtained by equating it to 0. i.e., $x + 1 = 0$, which gives $x = -1$. If $f(x)$ is a polynomial in x then $f(x) = 0$ is called a polynomial equation in x . We observe that '-1' is the root of the polynomial $f(x) = 0$ in the above example. So we say that '-1' is the zero of the polynomial $x + 1$, or a root of the polynomial equation $x + 1 = 0$.

- Now, consider the constant polynomial 3. Can you tell what is its zero ? It does not have a zero. As $3 = 3x^0$ no real value of x gives value of $3x^0$. Thus a constant polynomial has no zeroes. But zero polynomial is a constant polynomial having many zeros.

Try this

Find zeroes of the following polynomials

1. $2x - 3$
2. $x^2 - 5x + 6$
3. $x + 5$

Example-1. $p(x) = x + 2$. Find $p(1)$, $p(2)$, $p(-1)$ and $p(-2)$. Which among 1, 2, -1 and -2 becomes the 0 of $p(x)$?

Solution : Let $p(x) = x + 2$

replace x by 1

$p(1) = 1 + 2 = 3$

replace x by 2

$p(2) = 2 + 2 = 4$

replace x by -1

$p(-1) = -1 + 2 = 1$

replace x by -2

$p(-2) = -2 + 2 = 0$

Therefore, 1 , 2, -1 are not the zeroes of the polynomial $x + 2$, but -2 is the zero of the polynomial.

Example-2. Find zero of the polynomial $p(x) = 3x + 1$

Solution : Finding a zero of $p(x)$, is same as solving the equation

$$p(x) = 0$$

i.e. $$3x + 1 = 0$$

$$3x = -1$$

$$x = -\frac{1}{3}$$

So, $-\frac{1}{3}$ is a zero of the polynomial $3x+1$.

Example-3. Find zero of the polynomial $2x-1$.

Solution : Finding a zero of $p(x)$, is the same as solving the equation $p(x)=0$

Assume $p(x) = 2x - 1$, as $2x - 1 = 0$

$$x = \frac{1}{2} \text{ (how ?)}$$

Check it by finding the value of $P\left(\frac{1}{2}\right)$

2.4 (b) Zero of the Linear Polynomial in One Variable

Now, if $p(x) = ax+b$, $a \neq 0$, a linear polynomial, how you find a zero of $p(x)$?

As we have seen to find zero of a polynomial $p(x)$, we need to solve the polynomial equation $p(x) = 0$

Which means $ax+b=0$, $a \neq 0$

$$\text{So } ax = -b$$

$$\text{i.e., } x = \frac{-b}{a}$$

So, $x = \frac{-b}{a}$ is the only zero of the polynomial $p(x)=ax+b$ i.e., **A linear polynomial in one variable has only one zero.**

Do This

Fill in the blanks:

Linear Polynomial	Zero of the polynomial
$x + a$	$-a$
$x - a$	-------------
$ax + b$	-------------
$ax - b$	$\frac{b}{a}$

Example-4. Verify whether 2 and 1 are zeroes of the polynomial $x^2 - 3x + 2$ or not?

Solution : Let $p(x) = x^2 - 3x + 2$

replace x by 2

$$p(2) = (2)^2 - 3(2) + 2$$
$$= 4 - 6 + 2 = 0$$

also replace x by 1

$$p(1) = (1)^2 - 3(1) + 2$$
$$= 1 - 3 + 2$$
$$= 0$$

Hence, both 2 and 1 are zeroes of the polynomial $x^2 - 3x + 2$.

Is there any other way of checking this?

What is the degree of the polynomial $x^2 - 3x + 2$? Is it a linear polynomial ? No, it is a quadratic polynomial. Hence, a quadratic polynomial has two zeroes.

Example-5. If 3 is a zero of the polynomial $x^2 + 2x - a$, then find a.

Solution : Let $p(x) = x^2 + 2x - a$

As the zero of this polynomial is 3, we know that $p(3) = 0$.

$$x^2 + 2x - a = 0$$

Put $x = 3$, $(3)^2 + 2(3) - a = 0$

$$9 + 6 - a = 0$$
$$15 - a = 0$$
$$-a = -15$$

or $a = 15$

Think, Discuss and Write

1. $x^2 + 1$ has no real zeroes. Why?
2. Can you tell the number of zeroes of a polynomial of n^{th} degree?

Exercise - 2.2

1. Find the value of the polynomial $4x^2 - 5x + 3$, at

(i) $x = 0$ (ii) $x = -1$ (iii) $x = 2$ (iv) $x = \frac{1}{2}$

2. Find $p(0), p(1)$ and $p(2)$ for each of the following polynomials.

(i) $p(x) = x^2 - x + 1$ (ii) $p(y) = 2 + y + 2y^2 - y^3$

(iii) $p(z) = z^3$ (iv) $p(t) = (t-1)(t+1)$

(v) $p(x) = x^2 - 3x + 2$

3. Verify whether the values of x given in each case are the zeroes of the polynomial or not?.

(i) $p(x) = 2x + 1; \; x = -\frac{1}{2}$ (ii) $p(x) = 5x - \pi; \; x = \frac{-3}{2}$

(iii) $p(x) = x^2 - 1; \; x = \pm 1$ (iv) $p(x) = (x - 1)(x + 2); \; x = -1, -2$

(v) $p(y) = y^2; \; y = 0$ (vi) $p(x) = ax + b; \; x = -\frac{b}{a}$

(vii) $f(x) = 3x^2 - 1; \; x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}$ (viii) $f(x) = 2x - 1, \; x = \frac{1}{2}, \frac{-1}{2}$

4. Find the zero of the polynomial in each of the following cases.

(i) $f(x) = x + 2$ (ii) $f(x) = x - 2$ (iii) $f(x) = 2x + 3$

(iv) $f(x) = 2x - 3$ (v) $f(x) = x^2$ (vi) $f(x) = px, \; p \neq 0$

(vii) $f(x) = px + q, \; p \neq 0, \; p\, q$ are real numbers.

5. If 2 is a zero of the polynomial $p(x) = 2x^2 - 3x + 7a$, then find the value of a.

6. If 0 and 1 are the zeroes of the polynomial $f(x) = 2x^3 - 3x^2 + ax + b$, then find the values of a and b.

2.5 Dividing Polynomials

Observe the following examples

(i) Let us consider two numbers 25 and 3. Divide 25 by 3. We get the quotient 8 and remainder 1. We write

Dividend = (Divisor × Quotient) + Remainder

So, $25 = (8 \times 3) + 1$

Similarly divide 20 by 5, we get $20 = (4 \times 5) + 0$

Here the remainder is 0. In this case we say that 5 is a **factor** of 20 or 20 is a **multiple of 5**.

As we divide a number by another non-zero number, we can also divide a polynomial by another polynomial? Let's see.

(ii) Divide the polynomial $3x^3 + x^2 + x$ by the monomial x $(x \neq 0)$.

We have $(3x^3 + x^2 + x) \div x = \dfrac{3x^3}{x} + \dfrac{x^2}{x} + \dfrac{x}{x}$

$= 3x^2 + x + 1$

In fact x is a common factor to each term of $3x^3 + x^2 + x$ So we can write

$3x^3 + x^2 + x$ as $x(3x^2 + x + 1)$

What are the factors of $3x^3 + x^2 + x$?

(iii) Consider another example $(2x^2 + x + 1) \div x$ $(x \neq 0)$

$$\begin{array}{r|l} & \;\; 2x+1 \\ \hline x & 2x^2 + x + 1 \\ & -2x^2 \\ \hline & \quad\;\; x + 1 \\ & \quad\;\; -x \\ \hline & \qquad\; 1 \\ \hline \end{array}$$

Here, $(2x^2 + x + 1) \div x = \dfrac{2x^2}{x} + \dfrac{x}{x} + \dfrac{1}{x}$

$= 2x + 1 + \dfrac{1}{x}$

Is it a polynomial ?

As one of the term $\dfrac{1}{x}$ has a negative integer exponent (i.e. $\dfrac{1}{x} = x^{-1}$)

$\therefore$ $2x + 1 + \dfrac{1}{x}$ is not a polynomial.

We can however write

$(2x^2 + x + 1) = [x \times (2x + 1)] + 1$

By taking out 1 separately the rest of the polynomial can be written as product of two polynomials.

Here we can say $(2x + 1)$ is the quotient, x is the divisor and 1 is the remainder. We must keep in mind that since the remainder is not zero, 'x' is not a factor of $2x^2 + x + 1$.

Do This

1. Divide $3y^3 + 2y^2 + y$ by 'y' and write division fact $(y \neq 0)$.
2. Divide $4p^2 + 2p + 2$ by '$2p$' and write division fact $(p \neq 0)$.

Example-6. Divide $3x^2 + x - 1$ by $x + 1$.

Solution : Consider $p(x) = 3x^2 + x - 1$ and $q(x) = x + 1$.

Divide $p(x)$ by $q(x)$. Recall the division process you have learnt in earlier classes.

Step 1: Divide $\dfrac{3x^2}{x} = 3x$, it becomes first term in quotient.

Step 2 : Multiply $(x+1)\ 3x = 3x^2 + 3x$

by subtracting $3x^2 + 3x$ from $3x^2 + x$, we get $-2x$

Step 3 : Divide $\dfrac{-2x}{x} = -2$, it becomes the 2nd term in the quotient.

Step 4 : Multiply $(x+1)(-2) = -2x - 2$,

Subtract it from $-2x - 1$, which gives '1'.

Step 5 : We stop here as the remainder is 1, a constant.

(Can you tell why a constant is not divided by a polynomial?)

$$
\begin{array}{r|l}
 & 3x - 2 \\
\hline
x+1 & 3x^2 + x - 1 \\
 & 3x^2 + 3x \\
 & - \quad - \\
\hline
 & -2x - 1 \\
 & -2x - 2 \\
 & + \quad + \\
\hline
 & +1 \\
\hline
\end{array}
$$

This gives us the quotient as $(3x-2)$ and remainder $(+1)$.

Note : The division process is said to be complete if we get the remainder 0 or the degree of the remainder is less than the degree of the divisor.

Now, we can write the division fact as

$3x^2 + x - 1 = (x+1)(3x-2) + 1$

i.e. Dividend = (Divisor × Quotient) + Remainder.

Let us see by replacing x by -1 in $p(x)$

$p(x) \ = 3x^2 + x - 1$

$p(-1) = 3(-1)^2 + (-1) - 1$

$\quad = 3(+1) + (-1) - 1 = 1.$

It is observed that $p(-1)$ is same as the remainder '1' by obtained by actual division.

So, the remainder obtained on dividing $p(x) = 3x^2 + x - 1$ by $(x+1)$ is same as $p(-1)$ where -1 is the zero of $x+1$. i.e. $x = -1$.

Let us consider some more examples.

Example-7. Divide the polynomial $2x^4 - 4x^3 - 3x - 1$ by $(x-1)$ and verify the remainder with zero of the divisor.

Solution : Let $f(x) = 2x^4 - 4x^3 - 3x - 1$

First see how many times $2x^4$ is of x.

$$\frac{2x^4}{x} = 2x^3$$

Now multiply $(x-1)(2x^3) = 2x^4 - 2x^3$

Then again see the first term of the remainder that is $-2x^3$. Now do the same.

$$
\begin{array}{r|l}
 & 2x^3 - 2x^2 - 2x - 5 \\
\hline
x-1 & 2x^4 - 4x^3 - 3x - 1 \\
 & 2x^4 - 2x^3 \\
 & - \quad + \\
\hline
 & -2x^3 - 3x - 1 \\
 & -2x^3 + 2x^2 \\
 & + \quad - \\
\hline
 & -2x^2 - 3x - 1 \\
 & -2x^2 + 2x \\
 & + \quad - \\
\hline
 & -5x - 1 \\
 & -5x + 5 \\
 & + \quad - \\
\hline
 & -6 \\
\hline
\end{array}
$$

Here the quotient is $2x^3 - 2x^2 - 2x - 5$ and the remainder is -6.

Now, the zero of the polynomial $(x - 1)$ is 1.

Put $x = 1$ in $f(x)$, $f(x) = 2x^4 - 4x^3 - 3x - 1$

$$\begin{aligned} f(1) &= 2(1)^4 - 4(1)^3 - 3(1) - 1 \\ &= 2(1) - 4(1) - 3(1) - 1 \\ &= 2 - 4 - 3 - 1 \\ &= -6 \end{aligned}$$

Is the remainder same as the value of the polynomial $f(x)$ at zero of $(x - 1)$?

From the above examples we shall now state the fact in the form of the following theorem. It gives a remainder without actual division of a polynomial by a **linear polynomial in one variable**.

Remainder Theorem :

Let $p(x)$ be any polynomial of degree greater than or equal to one and let 'a' be any real number. If $p(x)$ is divided by the linear polynomial $(x - a)$, then the remainder is $p(a)$.

Let us now look at the proof of this theorem.

Proof : Let $p(x)$ be any polynomial with degree greater than or equal to 1.

Further suppose that when $p(x)$ is divided by a linear polynomial $g(x) = (x - a)$, the quotient is $q(x)$ and the remainder is $r(x)$. In other words, $p(x)$ and $g(x)$ are two polynomials such that the degree of $p(x) \geq$ degree of $g(x)$ and $g(x) \neq 0$ then we can find polynomials $q(x)$ and $r(x)$ such that, where $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$.

By division algorithm,

$$p(x) = g(x) \cdot q(x) + r(x)$$

$$\therefore\ p(x) = (x - a) \cdot q(x) + r(x) \qquad \because\ g(x) = (x - a)$$

Since the degree of $(x - a)$ is 1 and the degree of $r(x)$ is less than the degree of $(x - a)$.

$\therefore$ Degree of $r(x) = 0$, implies $r(x)$ is a constant, say K.

So, for every real value of x, $r(x) = \text{K}$.

Therefore,

$$p(x) = (x - a)\, q(x) + \text{K}$$

If $x = a$, then $p(a) = (a - a)\, q(a) + \text{K}$

$$\begin{aligned} &= 0 + \text{K} \\ &= \text{K} \end{aligned}$$

Hence proved.

Let us use this result in finding remainders when a polynomial is divided by a linear polynomial without actual division.

Example-8. Find the remainder when $x^3 + 1$ divided by $(x + 1)$

Solution : Here $p(x) = x^3 + 1$

The zero of the linear polynomial $x + 1$ is -1 $(x + 1 = 0 \Rightarrow x = -1)$

So replacing x by -1

$$\begin{aligned} p(-1) &= (-1)^3 + 1 \\ &= -1 + 1 \\ &= 0 \end{aligned}$$

So, by Remainder Theorem, we know that $(x^3 + 1)$ divided by $(x + 1)$ gives 0 as the remainder.

You can also check this by actual division method. i.e., $x^3 + 1$ by $x + 1$.

Can you say $(x + 1)$ is a factor of $(x^3 + 1)$?

Example-9. Check whether $(x - 2)$ is a factor of $x^3 - 2x^2 - 5x + 4$

Solution : Let $p(x) = x^3 - 2x^2 - 5x + 4$

To check whether the linear polynomial $(x - 2)$ is a factor of the given polynomial,

Replace x, by the zero of $(x - 2)$ i.e. $x - 2 = 0 \Rightarrow x = 2$.

$$\begin{aligned} p(2) &= (2)^3 - 2(2)^2 - 5(2) + 4 \\ &= 8 - 2(4) - 10 + 4 \\ &= 8 - 8 - 10 + 4 \\ &= -6. \end{aligned}$$

As the remainder is not equal to zero, the polynomial $(x - 2)$ is not a factor of the given polynomial $x^3 - 2x^2 - 5x + 4$.

Example10. Check whether the polynomial $p(y) = 4y^3 + 4y^2 - y - 1$ is a multiple of $(2y + 1)$.

Solution : As you know, $p(y)$ will be a multiple of $(2y + 1)$ only, if $(2y + 1)$ divides $p(y)$ exactly.

We shall first find the zero of the divisor , $2y + 1$, i.e., $y = \frac{-1}{2}$, Replace y by $\frac{-1}{2}$ in $p(y)$

$$p\left(\frac{-1}{2}\right) = 4\left(\frac{-1}{2}\right)^3 + 4\left(\frac{-1}{2}\right)^2 - \left(\frac{-1}{2}\right) - 1$$

$$= 4\left(\frac{-1}{8}\right) + 4\left(\frac{1}{4}\right) + \frac{1}{2} - 1$$

$$= \frac{-1}{2} + 1 + \frac{1}{2} - 1$$

$$= 0$$

So, $(2y + 1)$ is a factor of $p(y)$. That is $p(y)$ is a multiple of $(2y + 1)$.

Example-11. If the polynomials $ax^3 + 3x^2 - 13$ and $2x^3 - 5x + a$ are divided by $(x - 2)$ leave the same remainder, find the value of a.

Solution : Let $p(x) = ax^3 + 3x^2 - 13$ and $q(x) = 2x^3 - 5x + a$

$\because$ $p(x)$ and $q(x)$ when divided by $x - 2$ leave the same remainder.

$\therefore$ $p(2) = q(2)$

$$a(2)^3 + 3(2)^2 - 13 = 2(2)^3 - 5(2) + a$$

$$8a + 12 - 13 = 16 - 10 + a$$

$$8a - 1 = a + 6$$

$$8a - a = 6 + 1$$

$$7a = 7$$

$$a = 1$$

EXERCISE - 2.3

1. Find the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by the following Linear polynomials :

   (i) $x + 1$ (ii) $x - \frac{1}{2}$ (iii) x (iv) $x + \pi$

   (v) $5 + 2x$

2. Find the remainder when $x^3 - px^2 + 6x - p$ is divided by $x - p$.

3. Find the remainder when $2x^2 - 3x + 5$ is divided by $2x - 3$. Does it exactly divide the polynomial ? State reason.

4. Find the remainder when $9x^3 - 3x^2 + x - 5$ is divided by $x - \frac{2}{3}$

5. If the polynomials $2x^3 + ax^2 + 3x - 5$ and $x^3 + x^2 - 4x + a$ leave the same remainder when divided by $x - 2$, find the value of a.

6. If the polynomials $x^3 + ax^2 + 5$ and $x^3 - 2x^2 + a$ are divided by $(x + 2)$ leave the same remainder, find the value of a.

7. Find the remainder when $f(x) = x^4 - 3x^2 + 4$ is divided by $g(x) = x - 2$ and verify the result by actual division.

8. Find the remainder when $p(x) = x^3 - 6x^2 + 14x - 3$ is divided by $g(x) = 1 - 2x$ and verify the result by long division.

9. When a polynomial $2x^3 + 3x^2 + ax + b$ is divided by $(x - 2)$ leaves remainder 2, and $(x + 2)$ leaves remainder -2. Find a and b.

2.6 Factorising a polynomial

As we have already studied that a polynomial $q(x)$ is said to have divided a polynomial $p(x)$ exactly if the remainder is zero. In this case $q(x)$ is a factor of $p(x)$.

For example. When $p(x) = 4x^3 + 4x^2 - x - 1$ is divided by $g(x) = 2x + 1$, if the remainder is zero (verify)

then $4x^3 + 4x^2 - x - 1 = q(x)\,(2x + 1) + 0$

So $p(x) = q(x)\,(2x + 1)$

Therefore $g(x) = 2x + 1$ is a factor of $p(x)$.

With the help of Remainder Theorem can you state a theorem that helps to find the factors of a given polynomial ?

Factor Theorem : If $p(x)$ is a polynomial of degree $n \geq 1$ and 'a' is any real number, then (i) $x - a$ is a factor of $p(x)$, if $p(a) = 0$ (ii) and its converse "if $(x - a)$ is a factor of a polynomial $p(x)$ then $p(a) = 0$.

Let us see the simple proof of this theorem.

Proof : By Remainder Theorem,

$p(x) = (x - a)\; q(x) + p(a)$

(i) Consider proposition (i) If $p(a) = 0$, then $p(x) = (x - a)\, q(x) + 0.$

$= (x - a)\, q(x)$

Which shows that $(x-a)$ is a factor of $p(x)$.

Hence proved.

(ii) Consider proposition (ii) Since $(x-a)$ is a factor of $p(x)$, then $p(x) = (x-a)\,q(x)$ for some polynomial $q(x)$

$$\therefore \quad p(a) = (a-a)\,q(a) = 0$$

$\therefore$ Hence $p(a) = 0$ when $(x-a)$ is a factor of $p(x)$

Hence proved theorem.

Let us consider some more examples.

Let $p(x) = ax^3 + bx^2 + cx + d\ (a \neq 0)$ and $(x-1)$ is a factor of $p(x) \Rightarrow p(1) = 0$
$\Rightarrow a + b + c + d = 0$
i.e. the sum of the coefficients of a polynomial is zero then $(x-1)$ is a factor

Let $p(x) = ax^3 + bx^2 + cx + d\ (a \neq 0)$ and $(x+1)$ is a factor of $p(x) \Rightarrow p(-1) = 0$
$\Rightarrow b + d = a + c$
i.e. the sum of the coefficients of even power terms is equal to the sum of the coeffients odd power terms then $(x+1)$ is a factor.

Example-12. Examine whether $x+2$ is a factor of $x^3 + 2x^2 + 3x + 6$

Solution : Let $p(x) = x^3 + 2x^2 + 3x + 6$ and $g(x) = x + 2$

The zero of $g(x)$ is -2

Then
$$\begin{aligned} p(-2) &= (-2)^3 + 2(-2)^2 + 3(-2) + 6 \\ &= -8 + 2(4) - 6 + 6 \\ &= -8 + 8 - 6 + 6 \\ &= 0 \end{aligned}$$

So, by the Factor Theorem, $x+2$ is a factor of $x^3 + 2x^2 + 3x + 6$.

Example-13. Find the value of K, if $2x-3$ is a factor of $2x^3 - 9x^2 + x + K$.

Solution : $(2x-3)$ is a factor of $p(x) = 2x^3 - 9x^2 + x + K$,

If $(2x-3) = 0$ then $x = \frac{3}{2}$

$\therefore$ The zero of $(2x-3)$ is $\frac{3}{2}$

If $(2x-3)$ is a factor of $p(x)$ then $p\left(\frac{3}{2}\right) = 0$

$p(x) = 2x^3 - 9x^2 + x + K,$

$$\Rightarrow p\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + \frac{3}{2} + K = 0$$

$$\Rightarrow 2\left(\frac{27}{8}\right) - 9\left(\frac{9}{4}\right) + \frac{3}{2} + K = 0$$

$$\Rightarrow \left(\frac{27}{4} - \frac{81}{4} + \frac{3}{2} + K = 0\right) \times 4$$

$$27 - 81 + 6 + 4K = 0$$
$$-48 + 4K = 0$$
$$4K = 48$$
So $$K = 12$$

Example-14. Show that $(x-1)$ is a factor of $x^{10} - 1$ and also of $x^{11} - 1$.

Solution : Let $p(x) = x^{10} - 1$ and $g(x) = x^{11} - 1$

To prove $(x-1)$ is a factor of both $p(x)$ and $g(x)$, it is sufficient to show that $p(1) = 0$ and $g(1) = 0$.

Now

$p(x) = x^{10} - 1$ and $g(x) = x^{11} - 1$

$p(1) = (1)^{10} - 1$ and $g(1) = (1)^{11} - 1$

$= 1 - 1$ $\qquad$ $= 1 - 1$

$= 0$ $\qquad$ $= 0$

Thus by Factor Theorem,

$(x-1)$ is a factor of both $p(x)$ and $g(x)$.

TRY THIS

Show that $(x - 1)$ is a factor of $x^n - 1$.

We shall now try to factorise quadratic polynomial of the type $ax^2 + bx + c$, (where $a \neq 0$ and a, b, c are constants).

Let its factors be $(px + q)$ and $(rx + s)$.

Then $ax^2 + bx + c = (px + q)\ (rx + s)$

$$= prx^2 + (ps + qr)x + qs$$

By comparing the coefficients of x^2, x and constants, we get that,

$$a = pr$$

$$b = ps + qr$$

$$c = qs$$

This shows that b is the sum of two numbers ps and qr,

whose product is $(ps)\ (qr) = (pr)(qs)$

$$= ac$$

Therefore, to factorise $ax^2 + bx + c$, we have to write b as the sum of two numbers whose product is ac.

Example-15. Factorise $3x^2 + 11x + 6$

Solution : If we can find two numbers p and q such that $p + q = 11$ and $pq = 3 \times 6 = 18$, then we can get the factors.

So, let us see the pairs of factors of 18.

$(1, 18), (2, 9), (3, 6)$ of these pairs, 2 and 9 will satisfy $p + q = 11$

So $\quad 3x^2 + 11x + 6 \quad = 3x^2 + 2x + 9x + 6$

$$= x(3x + 2) + 3(3x + 2)$$

$$= (3x + 2)\,(x + 3).$$

Do this

Factorise the following

1. $6x^2 + 19x + 15$ $\qquad$ 2. $10m^2 - 31m - 132$ $\qquad$ 3. $12x^2 + 11x + 2$

Now, consider another example.

Example-16. Verify whether $2x^4 - 6x^3 + 3x^2 + 3x - 2$ is divisible by $x^2 - 3x + 2$ or not ?

How can you verify using Factor Theorem ?

Solution : The divisor is not a linear polynomial. It is a quadratic polynomial. You have learned the factorisation of a quadratic polynomial by splitting the middle term as follows.

$$x^2 - 3x + 2 \;= x^2 - 2x - x + 2$$

$$= x(x - 2) - 1(x - 2)$$

$$= (x - 2)\,(x - 1).$$

To show $x^2 - 3x + 2$ is a factor of polynomial $2x^4 - 6x^3 + 3x^2 + 3x - 2$, we have to show $(x - 2)$ and $(x - 1)$ are the factors of $2x^4 - 6x^3 + 3x^2 + 3x - 2$.

Let $\qquad p(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$

$\because$ $(x - 2)$ is factor of $p(x)$ then $p(2) = 2(2)^4 - 6(2)^3 + 3(2)^2 + 3(2) - 2$

$$= 2(16) - 6(8) + 3(4) + 6 - 2$$

$$= 32 - 48 + 12 + 6 - 2$$

$$= 50 - 50$$

$$= 0$$

As $p(2) = 0$, $(x - 2)$ is a factor of $p(x)$.

$(x-1)$ is to be another factor of $p(x)$

$$\begin{aligned} \text{Then} \quad p(1) &= 2(1)^4 - 6(1)^3 + 3(1)^2 + 3(1) - 2 \\ &= 2(1) - 6(1) + 3(1) + 3 - 2 \\ &= 2 - 6 + 3 + 3 - 2 \\ &= 8 - 8 \\ &= 0 \end{aligned}$$

As $p(1) = 0$, $(x-1)$ is a factor of $p(x)$.

As both $(x-2)$ and $(x-1)$ are factors of $p(x)$, then their product $x^2 - 3x + 2$ is also a factor of $p(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$.

- $(x-y) \mid (x^n - y^n)$, for all $n \in N$
- $(x+y) \mid (x^n - y^n)$, where n is even
- $(x+y) \mid (x^n + y^n)$, where n is odd
- $(x-y) \nmid (x^n + y^n)$, for all $n \in N$

Example-17. Factorise $x^3 - 23x^2 + 142x - 120$

Solution : Let $p(x) = x^3 - 23x^2 + 142x - 120$

By trial, we find that $p(1) = 0$. (verify)

So $(x-1)$ is a factor of $p(x)$

When we divide $p(x)$ by $(x-1)$, we get $x^2 - 22x + 120$.

Alternate method:

$$\begin{aligned} x^3 - 23x^2 + 142x - 120 &= x^3 - x^2 - 22x^2 + 22x + 120x - 120 \\ &= x^2(x-1) - 22x(x-1) + 120(x-1) \quad \text{(why?)} \\ &= (x-1)(x^2 - 22x + 120) \end{aligned}$$

Now $x^2 - 22x + 120$ is a quadratic expression that can be factorised by splitting the middle term. We have

$$\begin{aligned} x^2 - 22x + 120 &= x^2 - 12x - 10x + 120 \\ &= x(x-12) - 10(x-12) \\ &= (x-12)(x-10) \end{aligned}$$

So, $x^3 - 23x^2 + 142x - 120 = (x-1)(x-10)(x-12)$.

Note : $a \mid b$ (a divides b) means a is a factor b.

- $a \nmid b$ (a does not divide b) means a is not a factor of b.

EXERCISE - 2.4

1. Determine which of the following polynomials has $(x + 1)$ as a factor.

   (i) $x^3 - x^2 - x + 1$ (ii) $x^4 - x^3 + x^2 - x + 1$

   (iii) $x^4 + 2x^3 + 2x^2 + x + 1$ (iv) $x^3 - x^2 - (3 - \sqrt{3})\,x + \sqrt{3}$

2. Use the Factor Theorem to determine whether g(x) is factor of $f(x)$ in each of the following cases :

   (i) $f(x) = 5x^3 + x^2 - 5x - 1,\ g(x) = x + 1$

   (ii) $f(x) = x^3 + 3x^2 + 3x + 1,\ g(x) = x + 1$

   (iii) $f(x) = x^3 - 4x^2 + x + 6,\ g(x) = x - 2$

   (iv) $f(x) = 3x^3 + x^2 - 20x + 12,\ g(x) = 3x - 2$

   (v) $f(x) = 4x^3 + 20x^2 + 33x + 18,\ g(x) = 2x + 3$

3. Show that $(x - 2)$, $(x + 3)$ and $(x - 4)$ are factors of $x^3 - 3x^2 - 10x + 24$.

4. Show that $(x + 4)$, $(x - 3)$ and $(x - 7)$ are factors of $x^3 - 6x^2 - 19x + 84$.

5. If both $(x - 2)$ and $\left(x - \frac{1}{2}\right)$ are factors of $px^2 + 5x + r$, then show that $p = r$.

6. If $(x^2 - 1)$ is a factor of $ax^4 + bx^3 + cx^2 + dx + e$, then show that $a + c + e = b + d = 0$

7. Factorise (i) $x^3 - 2x^2 - x + 2$ (ii) $x^3 - 3x^2 - 9x - 5$

   (iii) $x^3 + 13x^2 + 32x + 20$ (iv) $y^3 + y^2 - y - 1$

8. If $ax^2 + bx + c$ and $bx^2 + ax + c$ have a common factor $x + 1$ then show that $c = 0$ and $a = b$.

9. If $x^2 - x - 6$ and $x^2 + 3x - 18$ have a common factor $(x - a)$ then find the value of a.

10. If $(y - 3)$ is a factor of $y^3 - 2y^2 - 9y + 18$ then find the other two factors.

2.7 ALGEBRAIC IDENTITIES

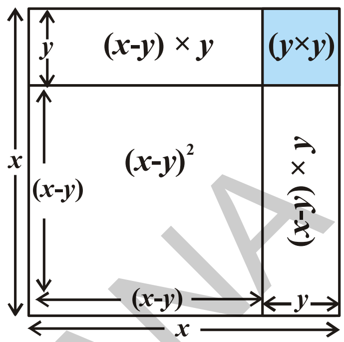
Recall that an algebraic Identity is an algebraic equation that is true for all values of the variables occurring in it. You have studied the following algebraic identities in earlier classes

Identity I : $(x + y)^2 \equiv x^2 + 2xy + y^2$

Identity II : $(x - y)^2 \equiv x^2 - 2xy + y^2$

Identity III : $(x + y)(x - y) \equiv x^2 - y^2$

Identity IV : $(x + a)(x + b) \equiv x^2 + (a + b)x + ab$.



Geometrical Proof :

For Identity $(x - y)^2$

Step-I Make a square of side x.

Step-II Subtract length y from x.

Step-III Calculate for $(x - y)^2$

$= x^2 - [(x - y)\,y + (x - y)\,y + y^2]$

$= x^2 - xy + y^2 - xy + y^2 - y^2$

$= x^2 - 2xy + y^2$

Try This

Try to draw the geometrical figures for other identities.

(i) $(x+y)^2 \equiv x^2 + 2xy + y^2$ (ii) $(x+y)(x-y) \equiv x^2 - y^2$

(iii) $(x+a)(x+b) \equiv x^2 + (a+b)x + ab$

(iv) $(x+a)(x+b)(x+c) \equiv x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc$

Do This

Find the following product using appropriate identities

(i) $(x + 5)(x + 5)$ (ii) $(p - 3)(p + 3)$ (iii) $(y - 1)(y - 1)$

(iv) $(t + 2)(t + 4)$ (v) 102×98 (vi) $(x + 1)(x + 2)(x + 3)$

Identities are useful in factorisation of algebraic expressions. Let us see some examples.

Example-18. Factorise

(i) $x^2 + 5x + 4$ (ii) $9x^2 - 25$

(iii) $25a^2 + 40ab + 16b^2$ (iv) $49x^2 - 112xy + 64y^2$

Solution :

(i) Here $x^2 + 5x + 4 = x^2 + (4 + 1)x + (4)(1)$

Comparing with Identity $(x + a)(x + b) \equiv x^2 + (a + b)x + ab$

we get $(x + 4)(x + 1)$.

(ii) $9x^2 - 25 = (3x)^2 - (5)^2$

Now comparing it with Identity III, $x^2 - y^2 \equiv (x + y)(x - y)$, we get

$\therefore 9x^2 - 25 = (3x + 5)(3x - 5)$.

(iii) Here you can see that

$25a^2 + 40ab + 16b^2 = (5a)^2 + 2(5a)\,(4b) + (4b)^2$

Comparing this expression with $x^2 + 2xy + y^2$,

we observe that $x = 5a$ and $y = 4b$

Using Identity I, $\boldsymbol{(x+y)^2 \equiv x^2 + 2xy + y^2}$

we get $25a^2 + 40ab + 16b^2 = (5a + 4b)^2$

$$= (5a + 4b)\,(5a + 4b).$$

- $(x + y)^2 + (x - y)^2 = 2(x^2 + y^2)$
- $(x + y)^2 - (x - y)^2 = 4xy$

(iv) Here $49x^2 - 112xy + 64y^2$, we see that

$49x^2 = (7x)^2$, $64y^2 = (8y)^2$ and

$112\,xy = 2(7x)\,(8y)$

Thus comparing with Identity II, $\boldsymbol{(x-y)^2 \equiv x^2 - 2xy + y^2}$,

we get, $49x^2 - 112xy + 64y^2 = (7x)^2 - 2(7x)\,(8y) + (8y)^2$

$$= (7x - 8y)^2$$

$$= (7x - 8y)\,(7x - 8y).$$

Do This

Factorise the following using appropriate identities

(i) $49a^2 + 70ab + 25b^2$ (ii) $\dfrac{9}{16}x^2 - \dfrac{y^2}{9}$

(iii) $t^2 - 2t + 1$ (iv) $x^2 + 3x + 2$

So far, all our identities involved products of binomials. Let us now extend the identity I to a trinomial $x + y + z$. We shall compute $(x + y + z)^2$.

Let $x + y = t$, then $(x + y + z)^2 = (t + z)^2$

$= t^2 + 2tz + z^2$ (using Identity I)

$= (x + y)^2 + 2(x + y)\,z + z^2$ (substituting the value of 't')

$= x^2 + 2xy + y^2 + 2xz + 2yz + z^2$

By rearranging the terms, we get $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

Alternate Method :

You can also compute $(x+y+z)^2$ by regrouping the terms

$$[(x+y)+z]^2 = (x+y)^2 + 2(x+y)(z) + (z)^2$$
$$= x^2 + 2xy + y^2 + 2xz + 2yz + z^2 \qquad \text{[From identity (1)]}$$
$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$$

In what other ways can you regroup the terms to find the expansion ? Will you get the same result ?

So, we can write the Identity as follows.

Identity V : $(x+y+z)^2 \equiv x^2+y^2+z^2+2xy+2yz+2zx$

Example-19. Expand $(2a+3b+5)^2$ using identity.

Solution : Comparing the given expression with $(x+y+z)^2$,

we find that $x = 2a$, $y = 3b$ and $z = 5$

Therefore, using Identity V, we have

$$(2a+3b+5)^2 = (2a)^2 + (3b)^2 + (5)^2 + 2(2a)(3b) + 2(3b)(5) + 2(5)(2a)$$
$$= 4a^2 + 9b^2 + 25 + 12ab + 30b + 20a.$$

Example-20. Find the product of $(5x-y+z)(5x-y+z)$

Solution : Here $(5x-y+z)(5x-y+z) = (5x-y+z)^2$

$$= [5x + (-y) + z]^2$$

Therefore using the Identity V, $(x+y+z)^2 \equiv x^2+y^2+z^2+2xy+2yz+2zx$, we get

$$(5x+(-y)+z)^2 = (5x)^2 + (-y)^2 + (z)^2 + 2(5x)(-y) + 2(-y)(z) + 2(z)(5x)$$
$$= 25x^2 + y^2 + z^2 - 10xy - 2yz + 10zx.$$

Example-21. Factorise $4x^2 + 9y^2 + 25z^2 - 12xy - 30yz + 20zx$

Solution : We have

$$4x^2 + 9y^2 + 25z^2 - 12xy - 30yz + 20zx$$
$$= [(2x)^2 + (-3y)^2 + (5z)^2 + 2(2x)(-3y) + 2(-3y)(5z) + 2(5z)(2x)]$$

Comparing with the identity V,

$(x + y + z)^2 \equiv x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$, we get

$$= (2x - 3y + 5z)^2$$

$$= (2x - 3y + 5z)\,(2x - 3y + 5z).$$

Do This

(i) Write $(p+2q+r)^2$ in expanded form.

(ii) Expand $(4x-2y-3z)^2$ using identity

(iii) Factorise $4a^2+b^2+c^2-4ab+2bc-4ca$ using suitable identity.

So far, we have dealt with identities involving second degree terms. Now let us extend Identity I to find $(x+y)^3$.

We have

$$\begin{aligned}(x+y)^3 &= (x+y)^2\,(x+y)\\ &= (x^2 + 2xy + y^2)\,(x+y)\\ &= x(x^2 + 2xy + y^2) + y(x^2 + 2xy + y^2)\\ &= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3\\ &= x^3 + 3x^2y + 3xy^2 + y^3\\ &= x^3 + 3xy\,(x+y) + y^3\\ &= x^3 + y^3 + 3xy(x+y).\end{aligned}$$

So, we get the following identity.

Identity VI : $(x+y)^3 \equiv x^3 + y^3 + 3xy\,(x+y)$.

Try This

How can you find $(x-y)^3$ without actual multiplication ?
Verify with actual multiplication.

You get the next identity as

Identity VII : $(x-y)^3 \equiv x^3 - y^3 - 3xy\,(x-y)$.
$\equiv x^3 - 3x^2y + 3xy^2 - y^3$

Let us see some examples where these identities are being used.

Example-22. Write the following cubes in the expanded form

(i) $(2a+3b)^3$ (ii) $(2p-5)^3$

Solution : (i) Comparing the given expression with $(x+y)^3$, we observe that $x = 2a$ and $y = 3b$

So, using Identity VI, we have

$$(2a + 3b)^3 = (2a)^3 + (3b)^3 + 3(2a)(3b)(2a + 3b)$$
$$= 8a^3 + 27b^3 + 18ab(2a + 3b)$$
$$= 8a^3 + 27b^3 + 36a^2b + 54ab^2$$
$$= 8a^3 + 36a^2b + 54ab^2 + 27b^3.$$

(ii) Comparing the given expression with $(x-y)^3$, we observe that $x = 2p$ and $y = 5$

So, using Identity VII, we have

$$(2p - 5)^3 = (2p)^3 - (5)^3 - 3(2p)(5)(2p - 5)$$
$$= 8p^3 - 125 - 30p(2p - 5)$$
$$= 8p^3 - 125 - 60p^2 + 150p$$
$$= 8p^3 - 60p^2 + 150p - 125.$$

Example-23. Evaluate each of the following using suitable identities

(i) $(103)^3$ (ii) $(99)^3$

Solution : (i) We have

$(103)^3 = (100 + 3)^3$

Comparing with $(x+y)^3 \equiv x^3 + y^3 + 3xy(x+y)$ we get

$$= (100)^3 + (3)^3 + 3(100)(3)(100 + 3)$$
$$= 1000000 + 27 + 900(103)$$
$$= 1000000 + 27 + 92700$$
$$= 1092727.$$

(ii) We have $(99)^3 = (100 - 1)^3$

Comparing with $(x-y)^3 \equiv x^3 - y^3 - 3xy(x-y)$ we get

$$= (100)^3 - (1)^3 - 3(100)(1)(100 - 1)$$
$$= 1000000 - 1 - 300(99)$$

$= 1000000 - 1 - 29700$

$= 970299.$

Example-24. Factorise $8x^3 + 36x^2y + 54xy^2 + 27y^3$.

Solution : The given expression can be written as

$$8x^3 + 36x^2y + 54xy^2 + 27y^3 = (2x)^3 + 3(2x)^2\,(3y) + 3(2x)\,(3y)^2 + (3y)^3$$

Comparing with Identity VI, $(x+y)^3 \equiv x^3 + 3x^2y + 3xy^2 + y^3$, we get

$$8x^3 + 36x^2y + 54xy^2 + 27y^3 = (2x+3y)^3$$

$= (2x + 3y)\,(2x + 3y)\,(2x + 3y)$ are factors.

Do This

1. Expand $(x+1)^3$ using an identity
2. Compute $(3m-2n)^3$.
3. Factorise $a^3 - 3a^2b + 3ab^2 - b^3$.

Now consider $(x+y+z)\,(x^2+y^2+z^2-xy-yz-zx)$

on expanding, we get the product as

$$= x(x^2+y^2+z^2-xy-yz-zx) + y(x^2+y^2+z^2-xy-yz-zx) + z(x^2+y^2+z^2-xy-yz-zx)$$

$$= x^3 + \cancel{xy^2} + \cancel{xz^2} - \cancel{x^2y} - xyz - \cancel{x^2z} + \cancel{x^2y} + y^3 + \cancel{yz^2} - \cancel{xy^2} - \cancel{y^2z} - xyz + \cancel{x^2z} + \cancel{y^2z} + z^3 - xyz - \cancel{yz^2} - \cancel{xz^2}$$

$= x^3 + y^3 + z^3 - 3xyz$ (on simplification)

Thus

$$\boxed{\textbf{Identity VIII : } (x+y+z)\,(x^2+y^2+z^2-xy-yz-xz) \equiv x^3+y^3+z^3-3xyz}$$

Example-25. Find the product

$$(2a + b + c)\,(4a^2 + b^2 + c^2 - 2ab - bc - 2ca)$$

Solution : Here the product that can be written as

$= (2a + b + c)\,[(2a)^2 + b^2 + c^2 - (2a)(b) - (b)(c) - (c)\,(2a)]$

Comparing with Identity VIII,

$(x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx) \equiv x^3 + y^3 + z^3 - 3xyz$

$= (2a)^3 + (b)^3 + (c)^3 - 3(2a)\,(b)\,(c)$

$= 8a^3 + b^3 + c^3 - 6abc$

Example-26. Factorise $a^3 - 8b^3 - 64c^3 - 24abc$

Solution : Here the given expression can be written as

$a^3 - 8b^3 - 64c^3 - 24abc = (a)^3 + (-2b)^3 + (-4c)^3 - 3(a)(-2b)(-4c)$

Comparing with the identity VIII,

$x^3 + y^3 + z^3 - 3xyz \equiv (x + y + z)\,(x^2 + y^2 + z^2 - xy - yz - zx)$

we get factors as

$= (a - 2b - 4c)\,[(a)^2 + (-2b)^2 + (-4c)^2 - (a)\,(-2b) - (-2b)\,(-4c) - (-4c)\,(a)]$

$= (a - 2b - 4c)\,(a^2 + 4b^2 + 16c^2 + 2ab - 8bc + 4ca).$

Do This

1. Find the product $(a - b - c)\,(a^2 + b^2 + c^2 - ab + bc - ca)$ without actual multiplication.
2. Factorise $27a^3 + b^3 + 8c^3 - 18abc$ using identity.

Example-27. Give possible values for length and breadth of the rectangle whose area is $2x^2 + 9x - 5$.

Solution : Let l, b be length and breadth of a rectangle

Area of rectangle $= 2x^2 + 9x - 5$

$lb = 2x^2 + 9x - 5$

$= 2x^2 + 10x - x - 5$

$= 2x(x + 5) - 1(x + 5)$

$= (x + 5)\,(2x - 1)$

$\therefore$ length $= (x + 5)$

breadth $= (2x - 1)$

Let $x = 1, \quad l = 6, \quad b = 1$

$x = 2, \quad l = 7, \quad b = 3$

$x = 3, \quad l = 8, \quad b = 5$

....

....

Can you find more values ?

Exercise - 2.5

1. Use suitable identities to find the following products

(i) $(x + 5)(x + 2)$ (ii) $(x - 5)(x - 5)$ (iii) $(3x + 2)(3x - 2)$

(iv) $\left(x^2 + \frac{1}{x^2}\right)\left(x^2 - \frac{1}{x^2}\right)$ (v) $(1 + x)(1 + x)$

2. Evaluate the following products without actual multiplication.

(i) 101×99 (ii) 999×999 (iii) $50\frac{1}{2} \times 49\frac{1}{2}$

(iv) 501×501 (v) 30.5×29.5

3. Factorise the following using appropriate identities.

(i) $16x^2 + 24xy + 9y^2$ (ii) $4y^2 - 4y + 1$

(iii) $4x^2 - \frac{y^2}{25}$ (iv) $18a^2 - 50$

(v) $x^2 + 5x + 6$ (vi) $3p^2 - 24p + 36$

4. Expand each of the following, using suitable identities

(i) $(x + 2y + 4z)^2$ (ii) $(2a - 3b)^3$ (iii) $(-2a + 5b - 3c)^2$

(iv) $\left(\frac{a}{4} - \frac{b}{2} + 1\right)^2$ (v) $(p + 1)^3$ (vi) $\left(x - \frac{2}{3}y\right)^3$

5. Factorise

(i) $25x^2 + 16y^2 + 4z^2 - 40xy + 16yz - 20xz$

(ii) $9a^2 + 4b^2 + 16c^2 + 12ab - 16bc - 24ca$

6. If $a + b + c = 9$ and $ab + bc + ca = 26$, then find $a^2 + b^2 + c^2$.

7. Evaluate the following using suitable identites.

(i) $(99)^3$ (ii) $(102)^3$ (iii) $(998)^3$ (iv) $(1001)^3$

8. Factorise each of the following

(i) $8a^3 + b^3 + 12a^2b + 6ab^2$ (ii) $8a^3 - b^3 - 12a^2b + 6ab^2$

(iii) $1 - 64a^3 - 12a + 48a^2$ (iv) $8p^3 - \frac{12}{5}p^2 + \frac{6}{25}p - \frac{1}{125}$

9. Verify (i) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ (ii) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

using some non-zero positive integers and check by actual multiplication. Can you call these as identites ?

10. Factorise (i) $27a^3 + 64b^3$ (ii) $343y^3 - 1000$ using the above results of problem (9).

11. Factorise $27x^3 + y^3 + z^3 - 9xyz$ using identity.

12. Verify that $x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x + y + z)[(x - y)^2 + (y - z)^2 + (z - x)^2]$

13. (a) If $x + y + z = 0$, show that $x^3 + y^3 + z^3 = 3xyz$.

(b) Show that $(a - b)^3 + (b - c)^3 + (c - a)^3 = 3(a - b)(b - c)(c - a)$

14. Without actual calculating the cubes, find the value of each of the following

(i) $(-10)^3 + (7)^3 + (3)^3$ (ii) $(28)^3 + (-15)^3 + (-13)^3$

(iii) $\left(\frac{1}{2}\right)^3 + \left(\frac{1}{3}\right)^3 - \left(\frac{5}{6}\right)^3$ (iv) $(0.2)^3 - (0.3)^3 + (0.1)^3$

15. Give possible expressions for the length and breadth of the rectangle whose area is given by

(i) $4a^2 + 4a - 3$ (ii) $25a^2 - 35a + 12$

16. What are the possible polynomial expressions for the dimensions of the cuboids whose volumes are given below?

(i) $3x^3 - 12x$ (ii) $12y^2 + 8y - 20$.

17. If $2(a^2+b^2) = (a+b)^2$, then show that $a = b$

WHAT WE HAVE DISCUSSED?

In this chapter, you have studied the following points.

1. A polynomial $p(x)$ in one variable x is an algebraic expression in x of the form
$p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots + a_2x^2 + a_1 x + a_0$, where $a_0, a_1, a_2, \ldots$. a_n are respectively the coefficients of $x^0, x^1, x^2, \ldots. x^n$ and n is called the **degree** of the polynomial if $a_n \neq 0$. Each a_nx^n ; $a_{n-1}x^{n-1}$; .... a_0, is called **a term** of the polynomial $p(x)$.
2. Polynomials are classified as monomial, binomial, trinomial etc. according to the number of terms in it.
3. Polynomials are also named as linear polynomial, quadratic polynomial, cubic polynomial etc. according to the degree of the polynomial.
4. A real number 'a' is a zero of a polynomial $p(x)$ if $p(a) = 0$. In this case, 'a' is also called a root of the polynomial equation $p(x) = 0$.
5. Every linear polynomial in one variable has a unique zero, a non-zero constant polynomial has no zero.
6. Remainder Theorem : If $p(x)$ is any polynomial of degree greater than or equal to 1 and $p(x)$ is divided by the linear polynomial $(x - a)$, then the remainder is $p(a)$.
7. Factor Theorem : If $x - a$ is a factor of the polynomial $p(x)$, then $p(a) = 0$. Also if $p(a) = 0$ then $(x - a)$ is a factor of $p(x)$.
8. Some Algebraic Identities are:
   (i) $(x + y + z)^2 \equiv x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$
   (ii) $(x + y)^3 \equiv x^3 + y^3 + 3xy(x + y)$
   (iii) $(x - y)^3 \equiv x^3 - y^3 - 3xy(x - y)$
   (iv) $x^3 + y^3 + z^3 - 3xyz \equiv (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$
   (v) $x^3 + y^3 \equiv (x + y)(x^2 - xy + y^2)$
   (vi) $x^3 - y^3 \equiv (x - y)(x^2 + xy + y^2)$
   (vii) $x^4 + 4y^4 = [(x + y)^2 + y^2][(x - y)^2 + y^2]$

Brain teaser

If $\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}} = \sqrt{x\sqrt{x\sqrt{x\ldots}}}$ then what is the value of x.

Chapter 3

The Elements of Geometry

J1Q9M4

3.1 Introduction

You may have seen large structures like bridges, dams, school buildings, hostels, hospitals etc. The construction of these structures is a big task for the engineers.

Do you know how we estimate the cost of the construction? Besides wages of the labour, cost of cement and concrete. It depends upon the size and shape of the structure.

The size and shape of a structure include the foundation, plinth area, size of the walls, elevation, roof etc. To understand the geometric principles involved in these constructions, we should know the basic elements of geometry and their applications.

We also know that geometry is widely used in daily life activities such as paintings, handicrafts, laying of floor designs, ploughing and sowing of seeds in fields. So in other words, we can say that the life without geometry is unimaginable.

The great construction like the Pyramids in Egypt, the Great wall of China, Temples, Mosques, Cathedral, Tajmahal, Charminar and altars in India, Eifel tower of France etc. are some of the best examples of application of geometry.

In this chapter, we will look into the history to understand the roots of geometry and the different schools of thought that have developed the geometry and its comparison with modern geometry.

3.2 History

The domains of mathematics which study the shapes and sizes of structures are described under geometry. The word *geometry* is derived from the Greek 'geo' means earth and 'metria' means measure.

The earliest recorded beginnings of geometry can be traced to early people, who discovered obtuse angled triangles in the ancient Indus valley and ancient Babylonia. The 'Bakshali manuscript' employs a handful of geometric problems including problems about volumes of irregular solids. Remnants of geometrical knowledge of the Indus Valley civilization can be found in excavations at Harappa and Mohenjo-Daro where there is evidence of circle-drawing instruments from as early as 2500 B.C.

The 'Sulabha Suthras' in Vedic Sanskrit lists the rules and geometric principles involved in the construction of ritual fire altars. The amazing idea behind the construction of fire altars is that they occupy same area although differ in their shapes. Boudhayana (8th century B.C.) composed the Boudhayana Sulabha Suthras, the best-known Sulabha Suthras which contains examples of simple Pythagorean triples such as (3,4,5), (5,12,13), (8,15,17)… etc. as well as a statement of Pythagorean theorem for the sides of a rectangle.

Ancient Greek mathematicians conceived ***geometry*** as the crown jewel of their sciences. They expanded the range of geometry to many new kinds of figures, curves, surfaces and solids. They found the need of establishing a proposed statement as universal truth with the help of logic. This idea led the Greek mathematician Thales to think of deductive proof.

Pythagoras of Ionia might have been a student of Thales and the theorem that was named after him might not have been his discovery, but he was probably one of the mathematicians who had given a deductive proof of it. Euclid (325-265B.C) of Alexandria in Egypt wrote 13 books called 'The Elements'. Thus Euclid created the first system of thought based on fundamental definitions, axioms, propositions and rules of inference through logic.

3.3 Euclid's Elements of Geometry

Euclid thought geometry as an abstract model of the world in which they lived. The notions of point, line, plane (or surface) and so on were derived from what was seen around them. From studies of the space and solids in the space around them, an abstract geometrical notion of a solid object was developed. A solid has shape, size, position and can be moved from one place to another. Its boundaries are called **surfaces**. They separate one part of the solid from another, and are said to have no thickness. The boundaries of the surfaces are **curves** or straight **lines**. These lines end in **points.** Consider the steps from solids to points (solids-surfaces-lines-point)

Observe the figure given in the next page. This figure is a cuboid (a solid) [fig.(i)]. It has three dimensions namely length, breadth and height. If it loses one dimension i.e. height then it will have only two dimensions which becomes a rectangle. You know that a rectangle has two dimensions length and breadth [fig.(ii)]. If it further loses another dimension i.e. breadth then it will leave with only line segment [fig.(iii)] and if it has to lose one more dimension, there remain only the points [fig.(iv)]. We may recall that a point has no dimensions. Similarly when we see the edge of a table or a book, we can visualise it as a line.

The end point of a line or the point where two lines meet is a point.

Fig

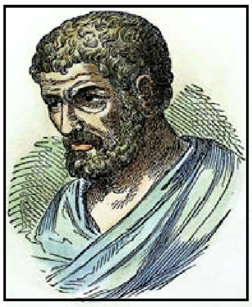
(i) (ii) (iii) (iv)

solids →	**surfaces/curves →**	**lines →**	**points**
3-D	2-D	1-D	no dimension

These are the fundamental terms of geometry. With the use of these terms other terms like line segment, angle, triangle etc. are defined.

From above these Euclid conformed all these statements as definitions.

Euclid defined point, line and plane in Book 1 of his Elements. Euclid listed 23 definitions. Some of them are given below. (Euclid's words)



Euclid 300 B.C
Father of Geometry

- A **point** is that which has no part
- A **line** is breadthless length
- The ends of a line are points
- A **straight line** is a line which lies evenly with the points on itself
- A **surface** is that which has length and breadth only
- The edges of surface are lines
- A **plane surface** is a surface which lies evenly with the straight lines on itself

In defining terms like point, line and plane, Euclid used words or phrases like 'part', 'breadth', 'evenly' which need **defining** or further explanation for the sake of clarity. In defining terms like plane, if we say 'a plane' occupies some area then 'area' is again to be clarified. So to define one term you need to define more than one term resulting in a chain of definitions without an end. So, mathematicians agreed to leave such terms as undefined. However we do have a intuitive feeling for the geometric concepts of a point than what the "definition" above gives us. So, we represent a point as a dot, even though a dot has some dimension. The Mohist philosophers in ancient China said "the line is divided into parts and that part which has no remaining part is a point.

A similar problem arises in definition 2 above, since it refers to breadth and length, neither of which has been defined. Because of this, a few terms are kept undefined while developing any course of study. So, in geometry, we *take a point, a line* *and a plane* (*in Euclid's words a plane surface*) *as undefined terms.* The only thing is that we can represent them intuitively, or explain them with the help of 'physical models.'

Euclid then used his definitions in assuming some geometric properties which need no proofs. These assumptions are self-evident truths. He divided them into two types: axioms and postulates.

3.3.1 Axioms and Postulates

Axioms are statements which are self evident or assumed to be true with in context of a particular mathematical system. For example when we say "The whole is always greater than the parts." It is a self evident fact and does not require any proof. This axiom gives us the definition of 'greater than'. For example, if a quantity P is a part of another quantity C, then C can be written as the sum of P and some third quantity R. Symbolically, $C > P$ means that there is some R such that $C = P + R$.

Euclid used this **common notion** or **axiom** throughout the mathematics not particularly in geometry but the term postulate was used for the assumptions made in geometry. The axioms are the foundation stones on which the structure of geometry is developed. These axioms arise in different situations. (Nowadays we are considering sometimes common notions and axioms as axioms.)

Some of the Euclid's axioms are given below.

- Things which are equal to the same things are equal to one another
- If equals are added to equals, the wholes are also equal
- If equals are subtracted from equals, the remainders are also equal.
- Things which coincide with one another are equal to one another.
- Things which are double of the same things are equal to one another
- Things which are halves of the same things are equal to one another

These 'common notions' refer to magnitudes of same kind. The first common notion could be applied to plane figures. For example, if the area of an object say A equals the area of another object B and the area of the object B equals that of a square, then the area of the object A is also equals to the area of the square.

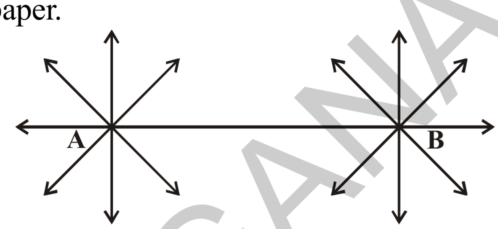
Magnitudes of the same kind can be compared and added, but magnitudes of different kinds cannot be compared. For example, a line cannot be added to the area of objects nor can an angle be compared to a pentagon.

Try This

Can you give any two axioms from your daily life.

Now let's discuss Euclid's five postulates:

1. Mark two distinct points A and B on a sheet of paper.



Draw a straight line passing through the points A and B. How many such lines can be drawn through point A and B? We can not draw more than one distinct line through two given points.

Euclid's first postulate gives the above concept. Postulate is as follows-

Postulate-1 : There is a unique line that passes through the given two distinct points.

In Euclid's terms, "To draw a straight line from any point to any point".

2. Draw a line segment PQ on a sheet of paper.

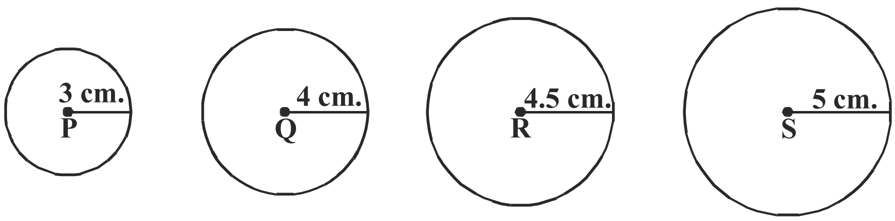
Extend the line segment both sides .

How far the line segment PQ can be extended both sides? Does it have any end points? W e see that the line segment PQ can be extended on both sides and the line PQ has no end points. Euclid conceived this idea in his second postulate.

Postulate-2 : A line segment can be extended on either side to form a line.

In Euclid's terms 'To produce a finite straight line continuously in a straight line' Euclid used the term 'terminated line' for 'a line segment'.

3. Radii of four circles are given as 3 cm, 4 cm, 4.5 cm and 5 cm. Using a compass, draw circles with these radii taking P, Q, R and S as their centres.



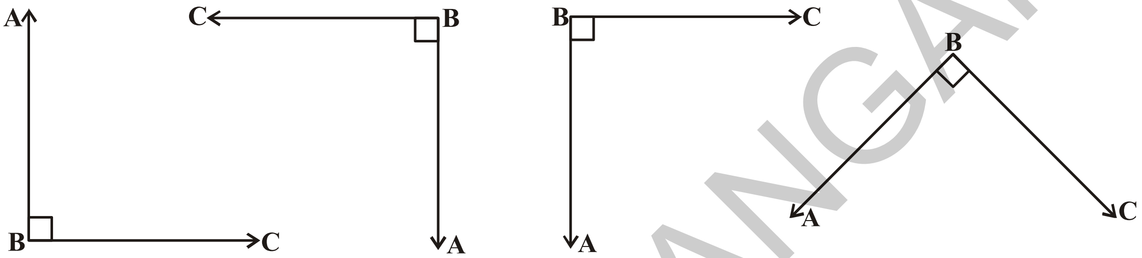
If the centre and radius of a circle are given can you draw the circle? We can draw a circle with any centre and any radius. (See chapter-12 Circle)

Euclid's third postulate states the above idea.

(To describe a circle with any centre and distance)

Postulate-3 : We can describe a circle with any centre and radius.

4. Take a grid paper. Draw different figures which represent a right angle. Cut them along their arms and place all angles one above other. What do you observe?



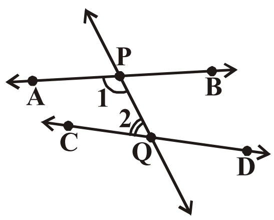
You observe that both the arms of each angle fall on one above the other, (i.e.) all right angles are equal. This is nothing but Euclids fourth axiom. Can you say this for any angle? Euclid has taken right angle as a reference angle for all the other angles and situation which he stated further.

Postulate-4 : All right angles are equal to one another.

Now we shall look at the Euclid's fifth postulate and its equivalent version.

Postulate-5 : That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

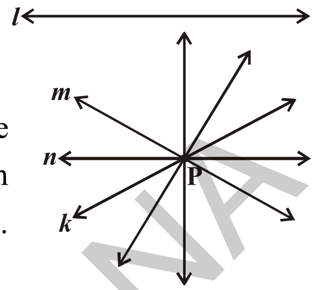
***Note* :** For example, the line PQ in figure falls on lines AB and CD such that the sum of the interior angles 1 and 2 is less than 180° on the left side of PQ. Therefore, the lines AB and CD will eventually intersect on the left side of PQ.



This postulate has acquired much importance as many mathematicians including Euclid were convinced that the fifth postulate is a theorem. Consequently for two thousand years mathematicians tried to prove that the fifth postulate was a consequence of Euclid's nine other axioms. They tried by assuming other proposition (John Play Fair) which are equivalent to it.

3.3.2 Equivalent version of fifth postulate or equivalents of fifth postulate

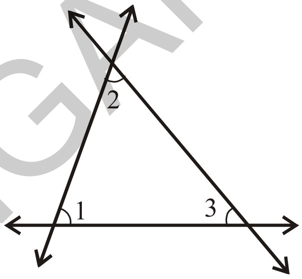
There are some noteworthy alternatives proposed by later mathematicians.



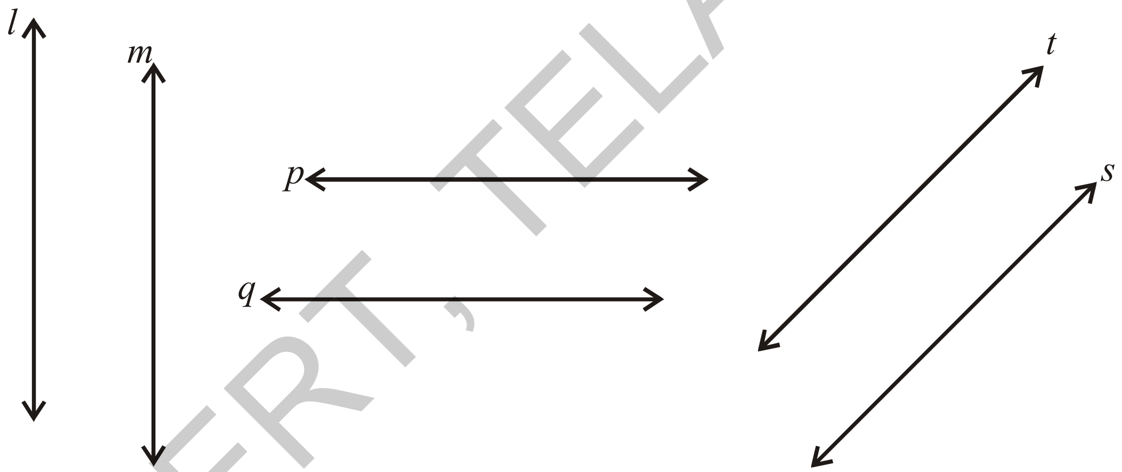
Through a point not on a given line, exactly one parallel line can be drawn to the given line. Let l be a line and P be a point, not on l. So through P, there exists only one line parallel to l. This is called Play Fair's axiom. (John Play Fair – 1748-1819)

- The sum of angles of any triangle is a constant and is equal to two right angles. (Legendre) i.e

$\angle 1 + \angle 2 + \angle 3 = 180^0$



- There exists a pair of lines everywhere equidistant from one another. (Posidominus)



- If a straight line intersects any one of two parallel lines, then it will intersect the other also.(Proclus)
- Straight lines parallel to the same straight line are parallel to one another.

If any one of these statements is substituted for Euclid's fifth postulate leaving the first four the same, the same geometry is obtained.

So after stating these five postulates, Euclid used them to prove many more results by applying deductive reasoning and the statements that were proved are called propositions or theorems.

Sometimes a certain statement that you think is to be true but that is a guess based on observations. Such statements which are neither proved nor disproved are called conjectures (hypothesis). Mathematical discoveries often start out as conjectures (hypothesis).

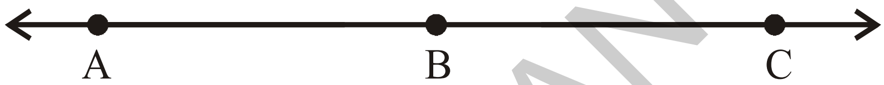
Gold Bach Conjecture : "Every even number greater than 4 can be written as sum of two primes" is a conjecture (hypothesis) stated by Gold Bach.

A conjecture (hypothesis) that is proved to be true is called a theorem. A theorem is proved by a logical chain of steps. A proof of a theorem is an argument that establishes the truth of the theorem beyond doubt.

Euclid deducted as many as 465 propositions in a logical chain using defined terms, axioms, postulates and theorems already proven in that chain.

Let us study how Euclid axioms and postulates can be used in proving the conjectures.

Example-1. If A, B, C are three points on a line and B lies between A and C, then prove that AC - AB = BC.



Solution : In the figure, AC coincides with AB+BC

Euclid's 4th axiom says that things which coincide with one another are equal to one another. Therefore it can be deduced that

$AB + BC = AC$,

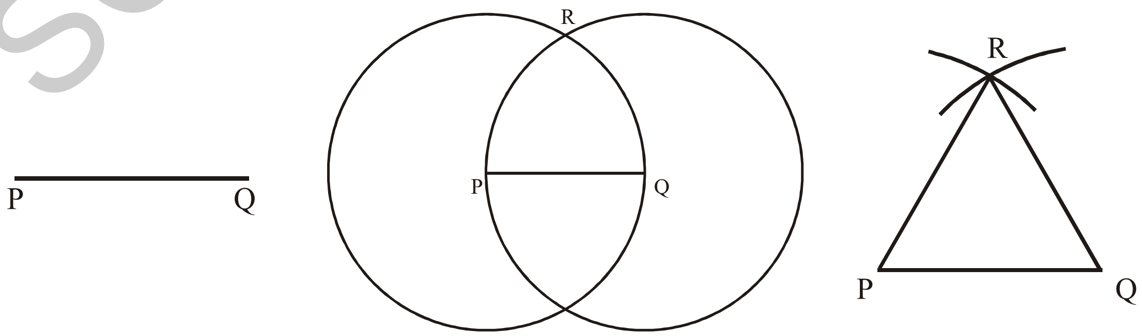
Substituting this value of AC in the given equation $AC - AB = BC$

$\cancel{AB} + BC - \cancel{AB} = BC$

Note that in this solution, it has been assumed that there is a unique line passing through two points.

Example-2. Prove that an equilateral triangle can be constructed on any given line segment.

Solution : It is given that; a line segment of any length say PQ



From Euclid's 3rd postulate, we can draw a circle with any centre and any radius. So, we can draw a circle with centre P and radius PQ. Draw another circle with centre Q and radius QP. The two circles meet at R. Join 'R' to P and R to Q to R form Δ PQR.

Now we require to prove the triangle thus formed is equilateral i.e., PQ = QR = RP.

PQ = PR (radii of the circle with centre P). Similarly, PQ = QR (radii of the circle with centre Q)

From Euclid's axiom, two things which are equal to same thing are equal to each another, we have PQ = QR = RP, so ΔPQR is an equilateral triangle. Note that here Euclid has assumed, without mentioning anywhere, that the two circles drawn with centre P and Q will meet each other at a point.

Let us now prove a theorem.

Example-3. Two distinct lines cannot have more than one point in common.

Given : *l* and m are given two lines.

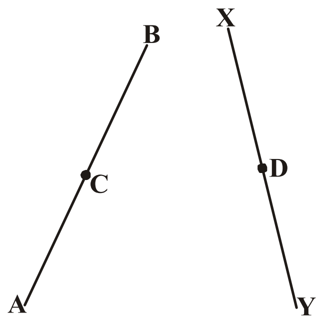
Required to Prove (RTP): *l* and m have only one point in common.

Proof: Let us assume that two lines intersect in two distinct points say A and B.

Now we have two lines passing through A and B. This assumption contradicts with the Euclid's axiom that only one line can pass through two distinct points. This contradiction arose due to our assumption that two lines can pass through two distinct points. So we can conclude that two distinct lines cannot have more than one point in common.

Note that Euclid state this for straight lines only, not for the curved lines. Where ever line is written always assume that we are talking about straight line.

Example-4. In the adjacent figure, we have AC = XD, C and D are mid points of AB and XY respectively. Show that AB = XY.



Solution : Given AB = 2 AC (C is mid point of AB)

XY = 2 XD (D is mid point of XY)

and AC = XD (given)

therefore, AB = XY

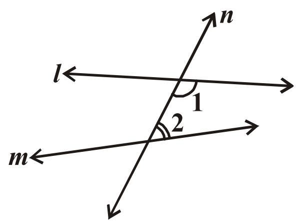
Since things which are double of the same things are equal to one another - Euclid "Common notion".

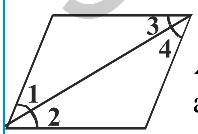
EXERCISE - 3.1

1. Answer the following:
   i. How many dimensions a solid has?
   ii. How many books are there in Euclid's Elements ?
   iii. Write the number of faces of cube and cuboid.
   iv. What is sum of interior angles of a triangle ?
   v. Write three un-defined terms of geometry.
2. State whether the following statements are true or false? Also give reasons for your answers.
   a) Only one line can pass through a given point.
   b) All right angles are equal.
   c) Circles with same radii are equal.
   d) A line segment can be extended on its both sides endlessly to get a straight line.

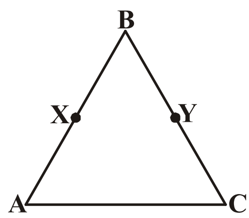
   e) From the figure, AB > AC
3. In the figure given below, show that length AH > AB + BC + CD.

4. If a point Q lies between two points P and R such that PQ = QR, prove that PQ = $\frac{1}{2}$ PR.
5. Draw an equilateral triangle whose sides are 5.2 cm. each.
6. What is a conjecture ? Give an example for it.
7. Mark two points P and Q. Draw a line through P and Q. Now how many lines which are parallel to PQ, can you draw?
8. In the adjacent figure, a line *n* falls on lines *l* and *m* such that the sum of the interior angles 1 and 2 is less than 180°, then what can you say about lines *l* and *m*.



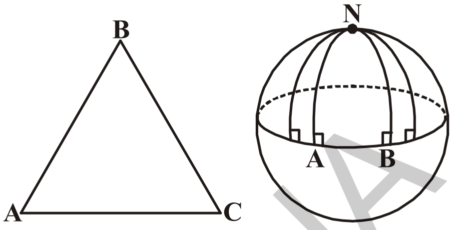


9. In the adjacent figure, if $\angle 1 = \angle 3$, $\angle 2 = \angle 4$ and $\angle 3 = \angle 4$, write the relation between $\angle 1$ and $\angle 2$ using an Euclid's postulate.
10. In the adjacent figure, we have BX = $\frac{1}{2}$ AB, BY = $\frac{1}{2}$ BC and AB = BC. Show that BX = BY



Non-Euclidian Geometry

The failure of attempts to prove the fifth postulate, gave new thoughts to Carl Fedrick Gauss, Lobachevsky and Bolyai. They thought fifth postulate is true or some contrary postulate can be substituted for it. If substituted with other, we obtain, Geometry different from Euclid's Geometry, hence called non-Euclidian Geometry.



(i) (ii)

If plane is not flat what happens to our theorems?

Let us observe.

Take a ball and try to draw a triangle on it? What difference do you find between triangle on plane and on a ball. You observe that lines of a triangle on plane are straight but not on ball.

See in figure (ii), the lines AN and BN (which are parts of great circles of a sphere) are perpendicular to the same line AB. But they are meeting at N, even though the sum of the angles on the same side of line AB is not less than two right angles (in fact, it is $90° + 90° = 180°$). Also, note that the sum of the angles of the triangle NAB on sphere is greater than $180°$, as $\angle A + \angle B = 180°$.

We call the plane on a sphere as a spherical plane. Can parallel lines exist on a sphere? Similarly by taking different planes and related axioms new geometries arise.

What we have discussed ?

- The three building blocks of geometry are Points, Lines and Planes, which are undefined terms.
- Ancient mathematicians including Euclid tried to define these undefined terms.
- Euclid developed a system of thought in his "The Elements" that serves as the foundation for development of all subsequent mathematics.
- Some of Euclid's axioms/common notation.
  - Things which are equal to the same things are equal to one another
  - If equals are added to equals, the wholes are also equal
  - If equals are subtracted from equals, the remainders are also equal.

I3A2X2

- Things which coincide with one another are equal to one another.
- The whole is greater than the part
- Things which are double of the same things are equal to one another
- Things which are halves of the same things are equal to one another

- Euclid's postulates are

Postulate - 1: To draw a straight line from any point to any point

Postulate - 2: A terminated line can be produced infinitely

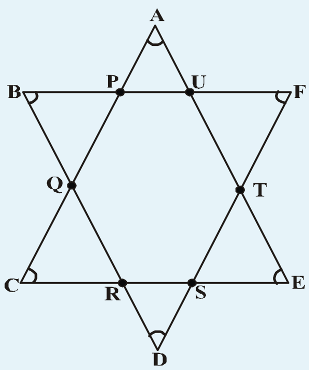
Postulate - 3: To describe a circle with any centre and radius

Postulate - 4: That all right angles equal to one another

Postulate - 5: That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the sum of the integer angles on the same side is less than two right angles.

Brain teaser

1. What is the measure of $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F$ in the figure given below. Give reason to your answer.



2. If the diagonal of a square is 'a' units, what is the diagonal of the square, whose area is double that of the first square?

Chapter

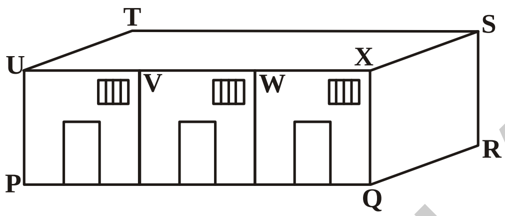
4

Lines and Angles

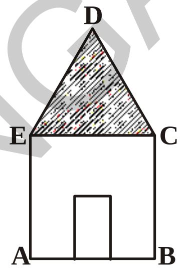
I5D5B8

4.1 Introduction

Reshma and Gopi have drawn the sketches of their school and home respectively. Can you identify some angles and line segments in these sketches?



(i)



(ii)

In the above figures (PQ, RS, ST, ...) and (AB, BC, CD, ...) are examples of line segments. Where as $\angle$QPU, $\angle$RQP, ... and $\angle$BAE, $\angle$CBA, ... are examples of some angles.

Do you know whenever an architect has to draw a plan for buildings, bridges, towers etc., the architect has to draw many lines and parallel lines at different angles.

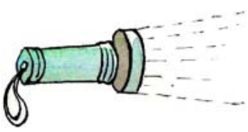
In Physics, we use lines and angles to assume and draw the movement of light and how the images are formed by reflection, refraction and scattering. Similarly while finding how much work is done by different forces acting on a body, we consider angles between force and displacement to find resultants. To find the height of a place we need both angles and lines. So in our daily life, we come across situations in which the basic ideas of geometry are in much use.

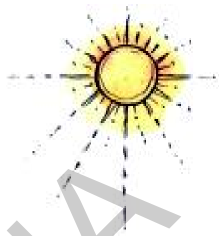
Do This

Observe your surroundings carefully and write any three situations of your daily life where you can observe lines and angles.

Draw these pictures in your note book and collect some pictures.

4.2 Basic Terms in Geometry





Think of a light beam originating from the sun or a torch light. How do you represent such a light beam? It's a ray starting from the sun. Recall that "a ray is a part of a line. "It begins at a point and goes on endlessly in a specified direction". While line can be extended in both directions endlessly.

A part of a line with two end points is known as line segment.

We usually denote a line segment AB by $\overline{AB}$ and its length is denoted by AB. The ray AB is denoted by $\overrightarrow{AB}$ and a line is denoted by $\overleftrightarrow{AB}$. However we normally use $\overleftrightarrow{AB}$, $\overleftrightarrow{PQ}$ for lines and some times small letters *l*, *m*, *n* etc. will also be used to denote lines.

If three or more points lie on the same line, they are called collinear points, otherwise they are called non-collinear points.

Sekhar marked some points on a line and try to count the line segments formed by them.

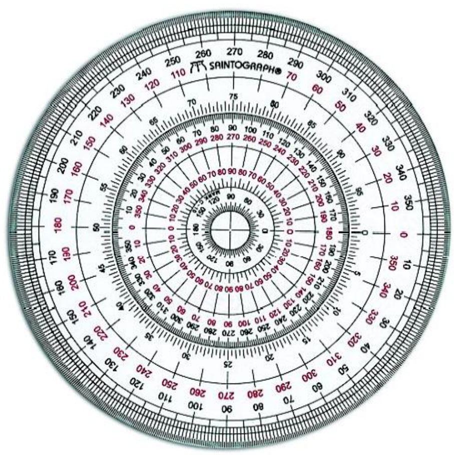
(**Note :** $\overline{PQ}$ and $\overline{QP}$ represents the same line segment)

S.No.	Points on line	Line Segments	Number
1.	P R Q	$\overline{PQ}$, $\overline{PR}$, $\overline{RQ}$	3
2.	P S R Q	$\overline{PQ}$, $\overline{PR}$, $\overline{PS}$, $\overline{SR}$, $\overline{SQ}$, $\overline{RQ}$	6
3.	P S T R Q	....	

Do you find any pattern between the number of points and line segments on a line?

Take some more points on the line and find the pattern:

No. of points on line	2	3	4	5	6	7
Total no. of line segments	1	3	6	.....	.....	.....

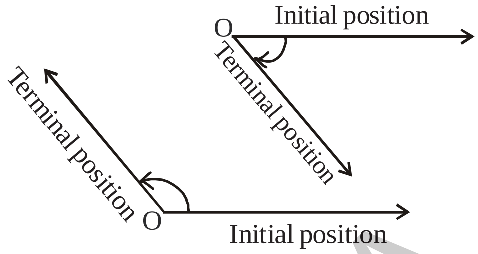


A circle is divided into 360 equal parts as shown in the figure.

The measure of angle of each part is called one degree.

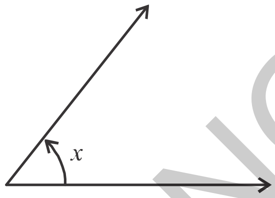
The angle is formed by rotating a ray from an initial position to a terminal position.

The change of a ray from initial position to terminal position around the fixed point 'O' is called rotation and measure of rotation is called angle.

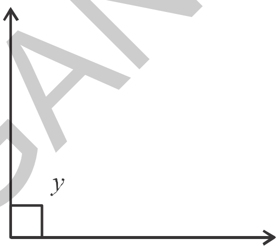


One complete rotation gives 360^0. We also draw angles with compass.

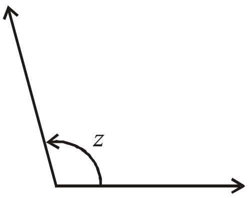
An angle is formed when two rays originate from the same point. The rays making an angle are called **arms** of the angle and the common point is called **vertex** of the angle. You have studied different types of angles, such as acute angle, right angle, obtuse angle, straight angle and reflex angle in your earlier classes.



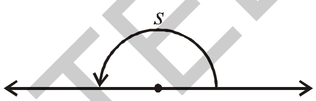
acute angle : $0° < x < 90°$



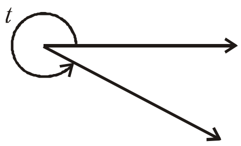
right angle : $y = 90°$



obtuse angle : $90° < z < 180°$



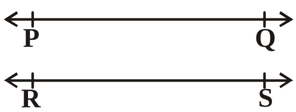
straight angle : $s = 180°$



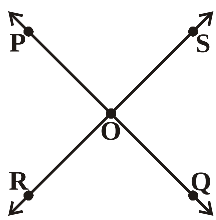
reflex angle : $180° < t < 360°$

4.2.1 Intersecting Lines and Non-intersecting Lines

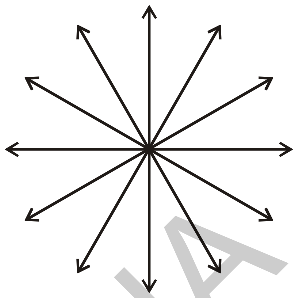
Observe the given figure. Do the lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ have any common points? What do we call such lines? They are called parallel lines.



On the other hand if they meet at any point, then they are called intersecting lines.



4.2.2 Concurrent Lines



How many lines can meet at a single point? Do you know the name of such lines? When three or more lines meet at a point, they are called concurrent lines and the point at which they meet is called point of concurrence.

THINK, DISCUSS AND WRITE

What is the difference between intersecting lines and concurrent lines?

EXERCISE - 4.1

1. In the given figure, name:

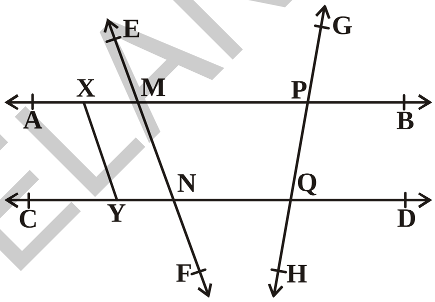
   (i) any six points

   (ii) any five line segments

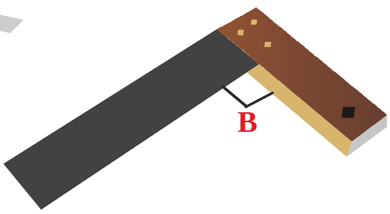
   (iii) any four rays

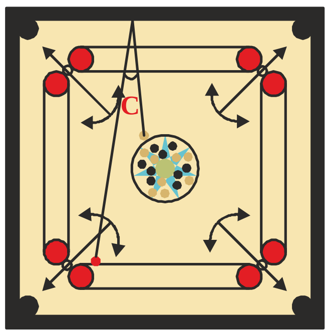
   (iv) any four lines

   (v) any four collinear points



2. Observe the following figures and identify the type of angles in them.





3. State whether the following statements are true or false :

   (i) A ray has no end point.

   (ii) Line $\overleftrightarrow{AB}$ is the same as line $\overleftrightarrow{BA}$.

   (iii) A ray $\overrightarrow{AB}$ is same as the ray $\overrightarrow{BA}$.

   (iv) A line has a definite length.

   (v) A plane has length and breadth but no thickness.

(vi) Two distinct points always determine a unique line.

(vii) Two lines may intersect in two points.

(viii) Two intersecting lines cannot both be parallel to the same line.

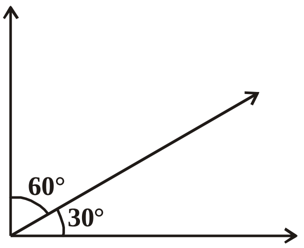
4. What is the angle between two hands of a clock when the time in the clock is

(a) 9'O clock (b) 6'O clock (c) 7:00 PM

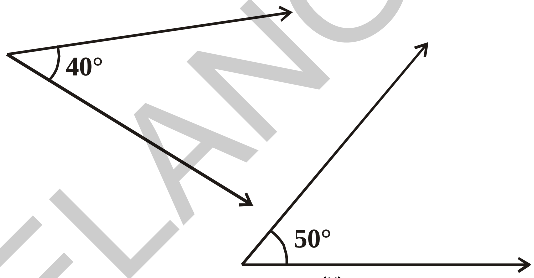
4.3 Pairs of Angles

Now let us discuss about some pairs of angles.

Observe the following figures and find the sum of angles.



(i)



(ii)

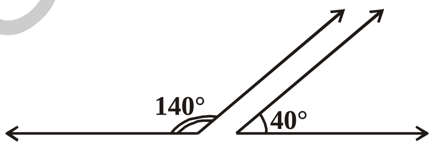
What is the sum of the two angles shown in each figure? It is 90^0. Do you know what do we call such pairs of angles? They are called complementary angles.

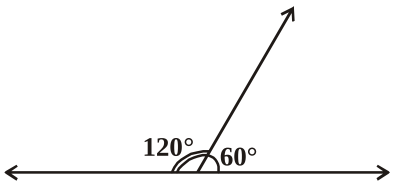
If a given angle is x^0, then what is its complementary angle? The complementary angle of x^0 is $(90^0 - x^0)$.

Example-1. If the measure of an angle is 62°, what is the measure of its complementary angle?

Solution : As the sum of comploementary angles is 90°, the complementary angle of 62° is 90° - 62° = 28°

Now observe the following figures and find the sum of angles in each figure.





What is the sum of the two angles shown in each figure? It is 180^0. Do you know what do we call such pair of angles? Yes, they are called supplementary angles. If the given angle is x^0, then what is its supplementary angle ? The supplementary angle of x^0 is $(180° - x°)$.

Example-2. Two complementary angles are in the ratio 4:5. Find the angles.

Solution : Let the required angles be $4x$ and $5x$.

Then $4x + 5x = 90^0$ (Why?)

$9x = 90^0 \Rightarrow x = 10^0$

Hence the required angles are 40^0 and 50^0.

Now observe the pairs of angles such as (120°, 240°) (100°, 260°) (180°, 180°) (50°, 310°) ..... etc. What do you call such pairs? The pair of angles, whose sum is 360° are called conjugate angles. Can you say the conjugate angle of 270°? What is the conjuage angle of x°?

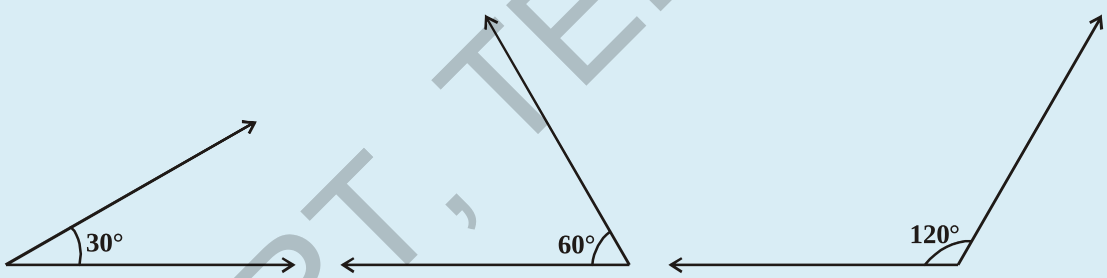
Do these

1. Write the complementary, supplementary and conjugate angles for the following angles.

   (a) 45^0 (b) 75^0 (c) 54° (d) 30^0

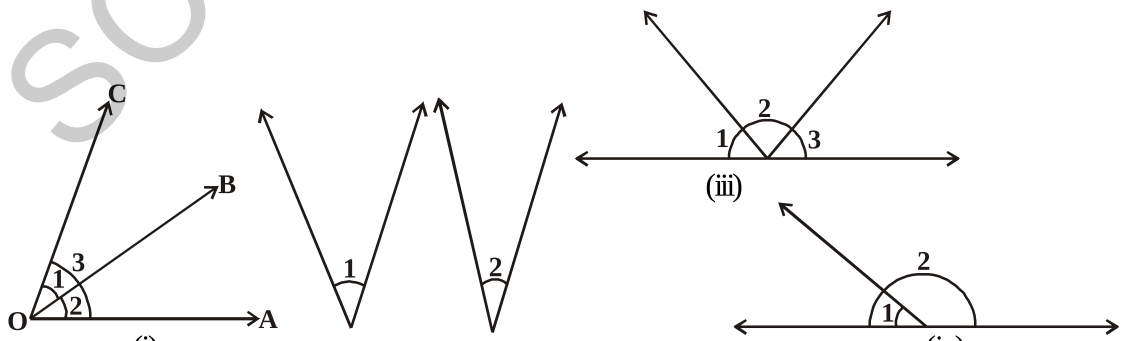
   (e) 60^0 (f) 90° (g) 0°

2. Which pairs of following angles are complementary or supplementary angles?



(i) (ii) (iii)

Observe the following figures, do they have any thing in common?



(i) (ii) (iii) (iv)

In figure (i) we can observe that vertex 'O' and arm '$\overrightarrow{OB}$' are common to both $\angle 1$ and $\angle 2$. What can you say about the non-common arms and how are they arranged? They are arranged on either side of the common arm. What do you call such pairs of angles?

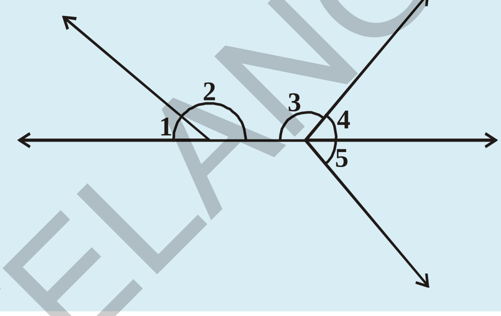
They are called a pair of adjacent angles.

In fig.(ii), two angles $\angle 1$ and $\angle 2$ are given. They have neither a common arm nor a common vertex. So they are not adjacent angles.

TRY THIS

(i) Find pairs of adjacent and non-adjacent angles in the above figures (i, ii, iii & iv).

(ii) Identify and write the adjacent angles in the given figure.



From the above, we can conclude that pairs of angles which have a common vertex, a common arm and non common arms lie on either side of common arm are called adjacent angles.

Observe the given figure. The hand of the athlete is making angles with the Javelin. What kind of angles are they? Obviously they are adjacent angles. Further what will be the sum of those two angles? Because they are on a straight line, the sum of the angles is 180^0. What do we call such pair of angles? They are called linear pair. So if the sum of two adjacent angles is 180^0, they are said to be a linear pair.

THINK, DISCUSS AND WRITE

Linear pair of angles are always supplementary. But supplementary angles need not form a linear pair. Why?

Activity

Measure the angles in the following figure and complete the table.

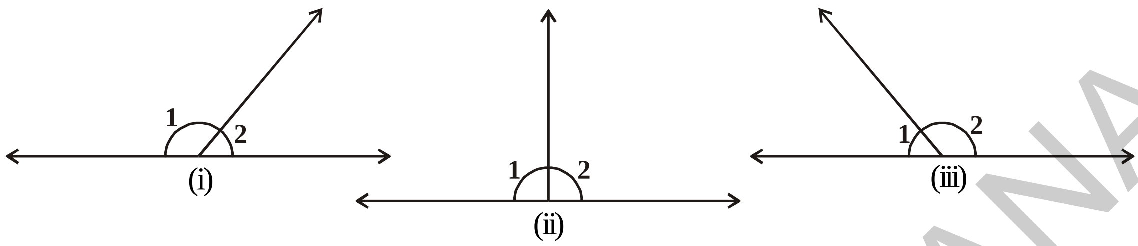
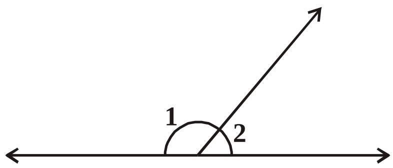


Figure	$\angle 1$	$\angle 2$	$\angle 1 + \angle 2$
(i)			
(ii)			
(iii)			

4.3.1 Linear pair of angles axiom

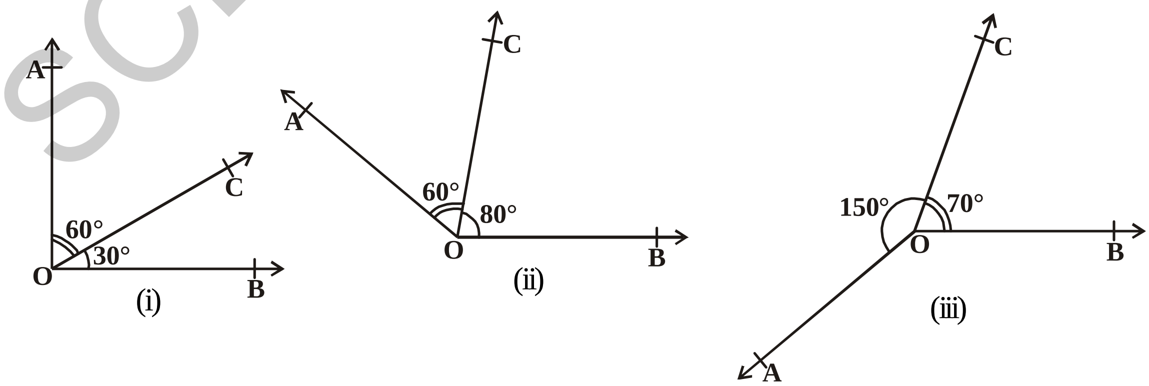
Axiom : If a ray stands on a straight line, then the sum of the two adjacent angles so formed is 180°.

When the sum of two adjacent angles is 180°, they are called a linear pair of angles.

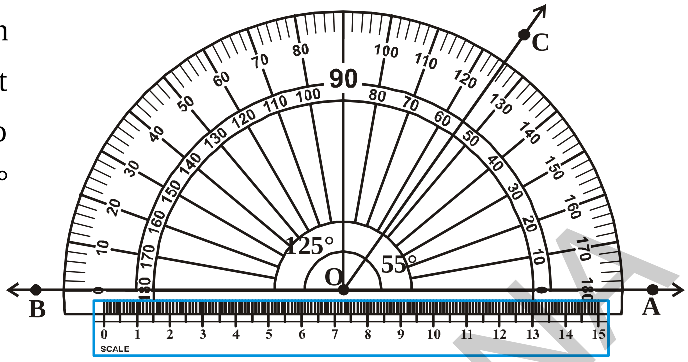


In the given figure, $\angle 1 + \angle 2 = 180^0$

Let us do the following. Draw adjacent angles of different measures as shown in the fig. Keep the ruler along one of the non-common arms in each case. Does the other non-common arm lie along the ruler?



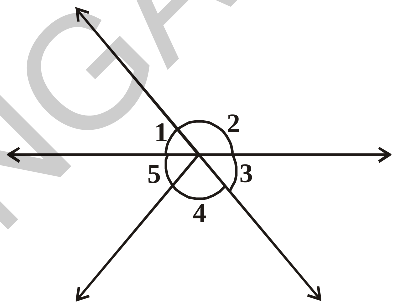
You will find that only in fig. (iv), both the non-common arms lie along the ruler, that is non common arms from a straight line. Also observe that $\angle AOC + \angle COB = 55° + 125° = 180°$. In other figures it is not so.



(iv)

Axiom : If the sum of two adjacent angles is 180°, then the non-common arms of the angles form a line. This is the converse of linear pair of angles axiom.

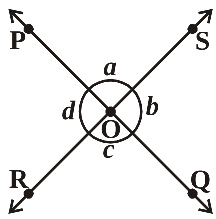
Angles at a point : We know that the sum of all the angles around a point is always 360°.



In the given figure $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 = 360^0$

4.3.2 Angles in intersecting lines

Draw any two intersecting lines and label them. Identify the linear pairs of angles and write down in your note book. How many linear pairs of angles are formed?

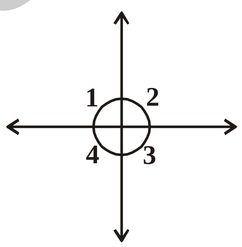


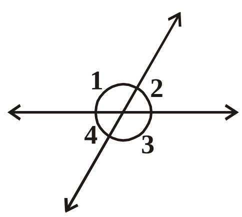
In the figure, $\angle POS$ and $\angle ROQ$ are opposite angles with same vertex and have no common arm. So they are called as **vertically opposite angles.**

How many pairs of vertically opposite angles are there? Can you find them? (See figure)

Activity

Measure the four angles 1, 2, 3, 4 in each of the below figure and complete the table:





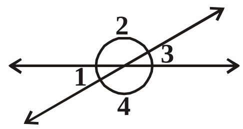


Figure	$\angle 1$	$\angle 2$	$\angle 3$	$\angle 4$
(i)				
(ii)				
(iii)				

What do you observe about the pairs of vertically opposite angles? Are they equal? Now let us prove this result in a logical way.

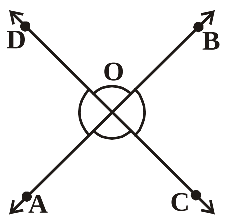
Theorem-4.1 : If two lines intersect each other, then the pairs of vertically opposite angles thus formed are equal.

Given: Let AB and CD be two lines intersecting at O

Required to prove (R.T.P.)

(i) $\angle AOC = \angle BOD$

(ii) $\angle DOA = \angle COB$.



Proof:

Ray $\overrightarrow{OA}$ stands on line $\overleftrightarrow{CD}$

Therefore, $\angle AOC + \angle DOA = 180°$ [Linear pair of angles axiom] .... (1)

Also $\angle DOA + \angle BOD = 180°$ [Why?] .... (2)

$\angle AOC + \angle DOA = \angle DOA + \angle BOD$ [From (1) and (2)]

$\angle AOC = \angle BOD$ [Cancellation of equal angles on both sides]

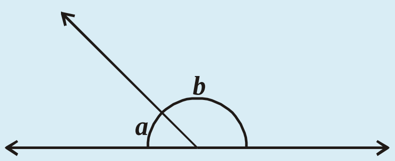
Similarly we can prove

$\angle DOA = \angle COB$

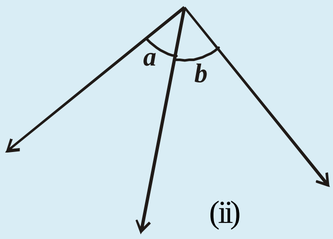
Do it on your own.

Do This

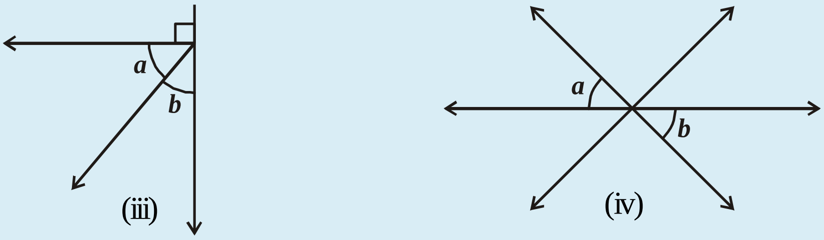
1. Classify the given angles as pairs of complementary, linear pair, vertically opposite and adjacent angles.



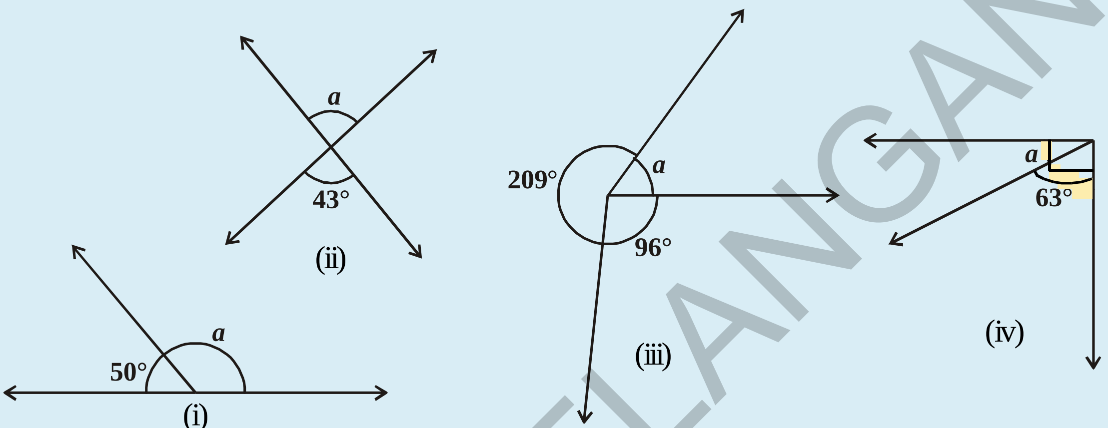
(i)



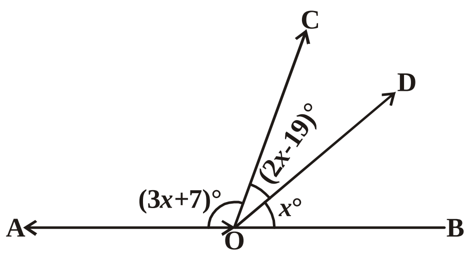
(ii)



2. Find the measure of angle ‘a’ in each figure. Give reason in each case.



Now, let us do some examples.



Example - 3. In the adjacent figure, $\overleftrightarrow{AB}$ is a straight line. Find the value of x and also find $\angle$AOC, $\angle$COD and $\angle$BOD.

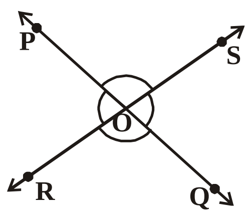
Solution : Since $\overleftrightarrow{AB}$ is a straight line, the sum of all the angles on $\overleftrightarrow{AB}$ at a point O is 180^0.

$$\therefore\ (3x + 7)^\circ + (2x - 19)^\circ + x^\circ = 180^\circ \text{ (Linear angles)}$$

$$\Rightarrow\ 6x^\circ - 12 = 180 \Rightarrow 6x^\circ = 192 \Rightarrow x^\circ = 32^\circ.$$

So, $\angle AOC = (3x + 7)^\circ = (3\times 32 + 7)^\circ = 103^\circ$,

$\angle COD = (2x - 19)^\circ = (2 \times 32 - 19)^\circ = 45^\circ$, $\angle BOD = 32^\circ$.



Example - 4. In the adjacent figure lines PQ and RS intersect each other at point O. If $\angle POR : \angle ROQ = 5 : 7$, find the measure of all the angles.

Solution : $\angle POR + \angle ROQ = 180^\circ$ (Linear pair of angles)

But $\angle POR : \angle ROQ = 5 : 7$ (Given)

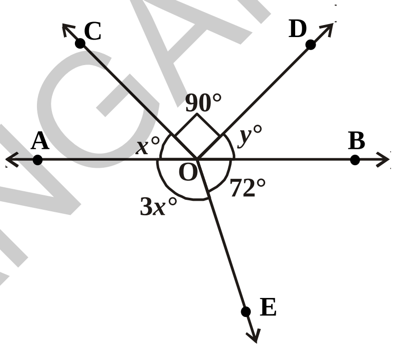
Therefore, $\angle POR = \frac{5}{12} \times 180 = 75°$

Similarly, $\angle ROQ = \frac{7}{12} \times 180 = 105°$

Now, $\angle POS = \angle ROQ = 105°$ (Vertically opposite angles)

and $\angle SOQ = \angle POR = 75°$ (Vertically opposite angles)

Example-5. Calculate $\angle AOC$, $\angle BOD$ and $\angle AOE$ in the adjacent figure given that $\angle COD = 90^o$, $\angle BOE = 72^o$ and AOB is a straight line,



Solution : Since AOB is a straight line, we have :

$\angle AOE + \angle EOB = 180^o$ (Linear pair)

$= 3x° + 72° = 180°$

$\Rightarrow 3x° = 108° \Rightarrow x° = 36°$.

From the figure

$\angle COA + \angle DOC + \angle BOD = 180^0$ ($\because$ straight angle)

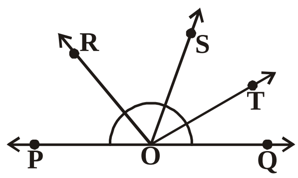
$\Rightarrow x° + 90° + y° = 180°$

$\Rightarrow 36° + 90° + y° = 180°$

$y° = 180° - 126° = 54°$

$\therefore \angle COA = 36^o$, $\angle BOD = 54^o$ and $\angle AOE = 108^o$.

Example-6. In the adjacent figure ray $\overrightarrow{OS}$ stands on a line $\overleftrightarrow{PQ}$. Ray $\overrightarrow{OR}$ and ray $\overrightarrow{OT}$ are angle bisectors of $\angle SOP$ and $\angle QOS$ respectively. Find $\angle TOR$.



Solution : Ray $\overrightarrow{OS}$ stands on the line $\overleftrightarrow{PQ}$.

Therefore, $\angle SOP + \angle QOS = 180°$ (Linear pair)

Let $\angle SOP = x°$

$\therefore x° + \angle QOS = 180°$ (How?)

So, $\angle QOS = 180° - x°$

Now, ray $\overrightarrow{OR}$ bisects $\angle SOP$

$\therefore \angle SOR = \frac{1}{2} \times \angle SOP$

$= \frac{1}{2} \times x° = \frac{x°}{2}$

Similarly, $\angle TOS = \frac{1}{2} \times \angle QOS$

$= \frac{1}{2} \times (180° - x°)$

$= 90° - \frac{x°}{2}$

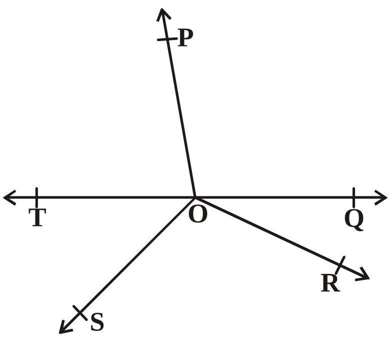
Now, $\angle TOR = \angle SOR + \angle TOS$

$= \frac{x°}{2} + \left(90° - \frac{x°}{2}\right)$

$= 90°$

Example-7. In the adjacent figure $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$ and $\overrightarrow{OS}$ are four rays. Prove that

$\angle QOP + \angle ROQ + \angle SOR + \angle POS = 360°$.



Solution : In the given figure, you need to draw opposite ray to any of the rays $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$ or $\overrightarrow{OS}$

Draw ray $\overrightarrow{OT}$ so that $\overleftrightarrow{TOQ}$ is a line. Now, ray OP stands on line $\overleftrightarrow{TQ}$.

$\therefore \angle TOP + \angle POQ = 180°$ .... (1) (Linear pair)

Similarly, ray $\overrightarrow{OS}$ stands on line $\overleftrightarrow{TQ}$.

$\therefore \angle TOS + \angle SOQ = 180°$ .... (2) (why?)

But $\angle SOQ = \angle SOR + \angle ROQ$

So, equation (2) becomes

$\angle TOS + \angle SOR + \angle ROQ = 180°$ .... (3)

Now, adding equations (1) and (3), you get

$\angle POT + \angle QOP + \angle TOS + \angle SOR + \angle ROQ = 360°$ .... (4)

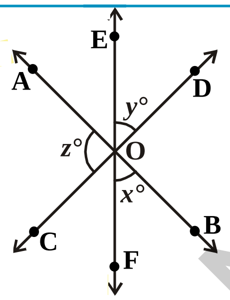
But $\angle POT + \angle TOS = \angle POS$

Therefore, equation (4) becomes

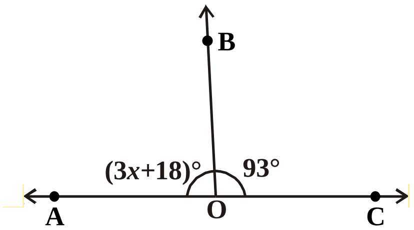
$\angle QOP + \angle ROQ + \angle SOR + \angle POS = 360°$

EXERCISE - 4.2

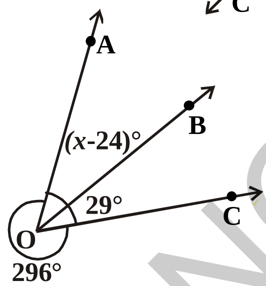
1. In the given figure three lines $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ intersecting at O. Find the values of x, y and z it is being given that x : y : z = 2 : 3 : 5



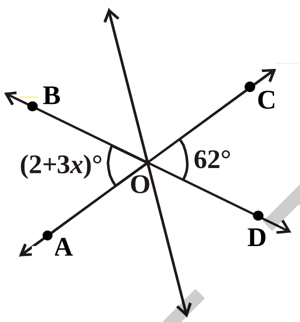
2. Find the value of x in the following figures.



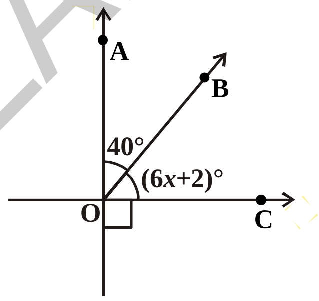
(i)



(ii)

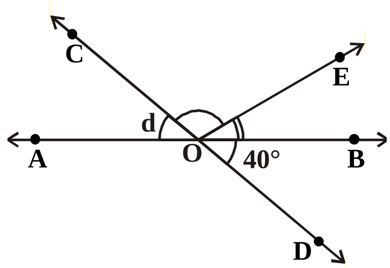


(iii)

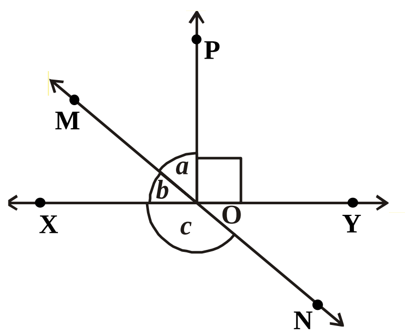


(iv)

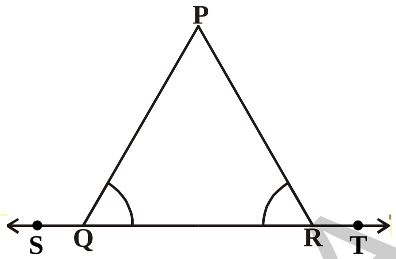
3. In the given figure lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O. If $\angle$AOC + $\angle$BOE = 70° and $\angle$DOB = 40°, find $\angle$BOE and reflex $\angle$EOC.



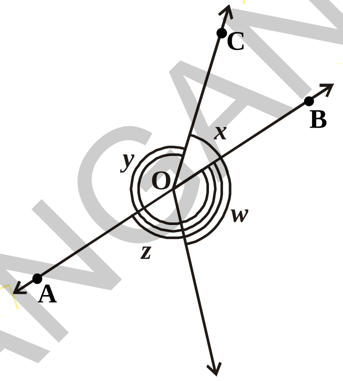
4. In the given figure lines $\overleftrightarrow{XY}$ and $\overleftrightarrow{MN}$ intersect at O. If $\angle$YOP = 90° and a: b = 2 : 3, find measure of c.



5. In the given figure $\angle RQP = \angle PRQ$, then prove that $\angle PQS = \angle TRP$.

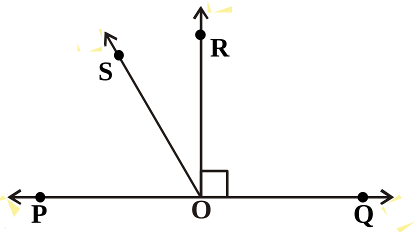


6. In the given figure, if $x + y = w + z$, then prove that AOB is a line.



7. In the given figure $\overleftrightarrow{PQ}$ is a line. Ray $\overrightarrow{OR}$ is perpendicular to line $\overleftrightarrow{PQ}$. $\overrightarrow{OS}$ is another ray lying between rays $\overrightarrow{OP}$ and $\overrightarrow{OR}$.

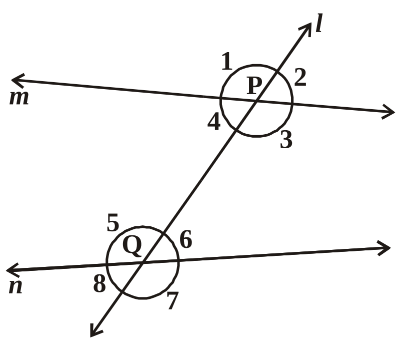
Prove that $\angle ROS = \frac{1}{2}(\angle QOS - \angle SOP)$



8. It is given that $\angle XYZ = 64°$ and XY is produced to point P. A ray YQ bisects $\angle ZYP$. Draw a figure from the given information. Find measures of $\angle XYQ$ and reflex $\angle QYP$.

4.4 Lines and a Transversal

Observe the figure. At how many points the line l meets the other lines m and n? Line l meets the lines at two distinct points. What do we call such a line? It is a transversal. It is a line which intersects two distinct lines at two distinct points. Line 'l' intersects lines 'm' and 'n' at points 'P' and 'Q' respectively. So, line l is a transversal for lines m and n.



Observe the number of angles formed when a transversal intersects a pair of lines.

If a transversal intersects two lines then eight angles will be formed.

Let us name these angles as $\angle 1$, $\angle 2$. . . $\angle 8$ as shown in the figure. Can you classify these angles? Some angles are exterior and some are interior. $\angle 1$, $\angle 2$, $\angle 7$ and $\angle 8$ are called exterior angles, while $\angle 3$, $\angle 4$, $\angle 5$ and $\angle 6$ are called interior angles.

The angles which are non-adjacent and lie on the same side of the transversal of which one is interior and the other is exterior, are called corresponding angles.

From the given figure.

(a) What are corresponding angles?

(i) $\angle 1$ and $\angle 5$ (ii) $\angle 2$ and $\angle 6$

(iii) $\angle 4$ and $\angle 8$ (iv) $\angle 3$ and $\angle 7$, So there are 4 pairs of corresponding angles.

(b) What are alternate interior angles?

(i) $\angle 4$ and $\angle 6$ (ii) $\angle 3$ and $\angle 5$, are two pairs of alternate interior angles.(Why?)

(c) What are alternate exterior angles?

(i) $\angle 1$ and $\angle 7$ (ii) $\angle 2$ and $\angle 8$, are two pairs of alternate exterior angles. (Why?)

(d) What are interior angles on the same side of the transversal?

(i) $\angle 4$ and $\angle 5$ (ii) $\angle 3$ and $\angle 6$ are two pairs of interior angles on the same side of the transversal. (Why?)

Interior angles on the same side of the transversal are also referred to as **consecutive interior angles** or **co-interior angles** or **allied interior angles**.

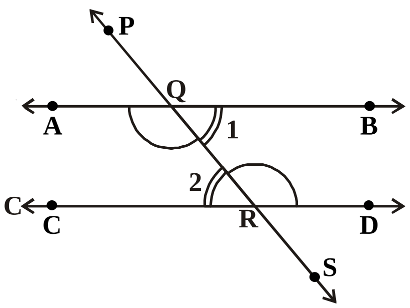
(e) What are exterior angles on the same side of the transversal?

(i) $\angle 1$, $\angle 8$ (ii) $\angle 2$, $\angle 7$ are two pairs of exterior angles on the same side of the transversal. (Why?)

Exterior angles on the same side of the transversal are also referred as consecutive exterior angle or co-exterior angles or allied exterior angles?

What can we say about the corresponding angles formed when the two lines l and m are parallel? Check and find. Are they equal? Yes, they are equal.

Axiom of corresponding angles: If a transversal intersects a pair of parallel lines, then each pair of corresponding angles are equal.



What is the relation between the pairs of alternate interior angles (i) $\angle RQB$ and $\angle QRC$

(ii) $\angle AQR$ and $\angle DRQ$ in the figure?

Can we use corresponding angles axiom to find the relation between these alternative interior angles.

In the figure, the transversal $\overleftrightarrow{PS}$ intersects two parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ at points Q and R respectively.

Let us prove $\angle RQB = \angle QRC$ and $\angle AQR = \angle DRQ$

You know that $\angle PQA = \angle QRC$ ..... (1) (corresponding angles axiom)

and $\angle PQA = \angle RQB$ ..... (2) (Why?)

So, from (1) and (2), you may conclude that $\angle RQB = \angle QRC$.

Similarly, $\angle AQR = \angle DRQ$.

This result can be stated as a theorem as follows:

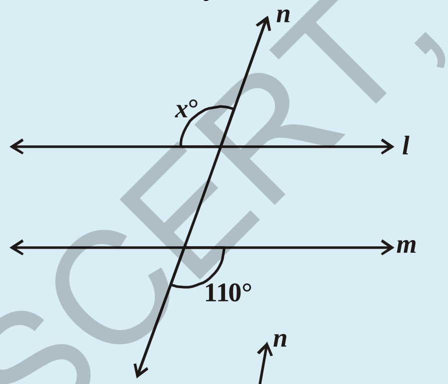
Theorem-4.2 : If a transversal intersects two parallel lines, then each pair of alternate interior angles are equal.

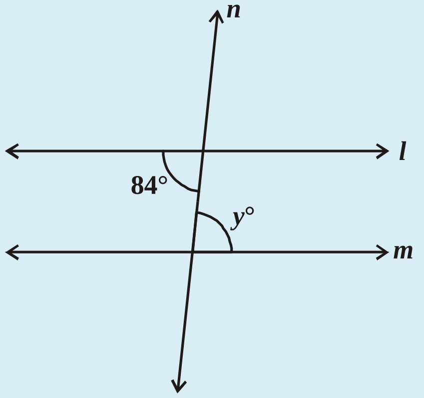
In a similar way, you can obtain the following theorem related to interior angles on the same side of the transversal.

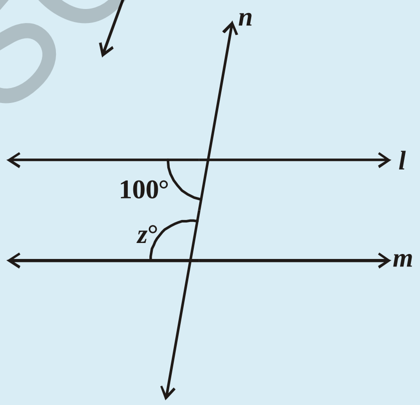
Theorem-4.3 : If a transversal intersects two parallel lines, then each pair of interior angles on the same side of the transversal are supplementary.

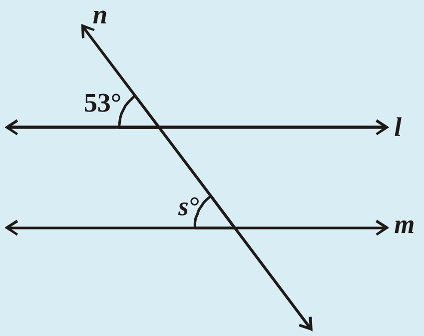
Do These

1. Find the measure of each angle indicated in each figure where *l* and *m* are parallel lines intersected by transversal *n*.

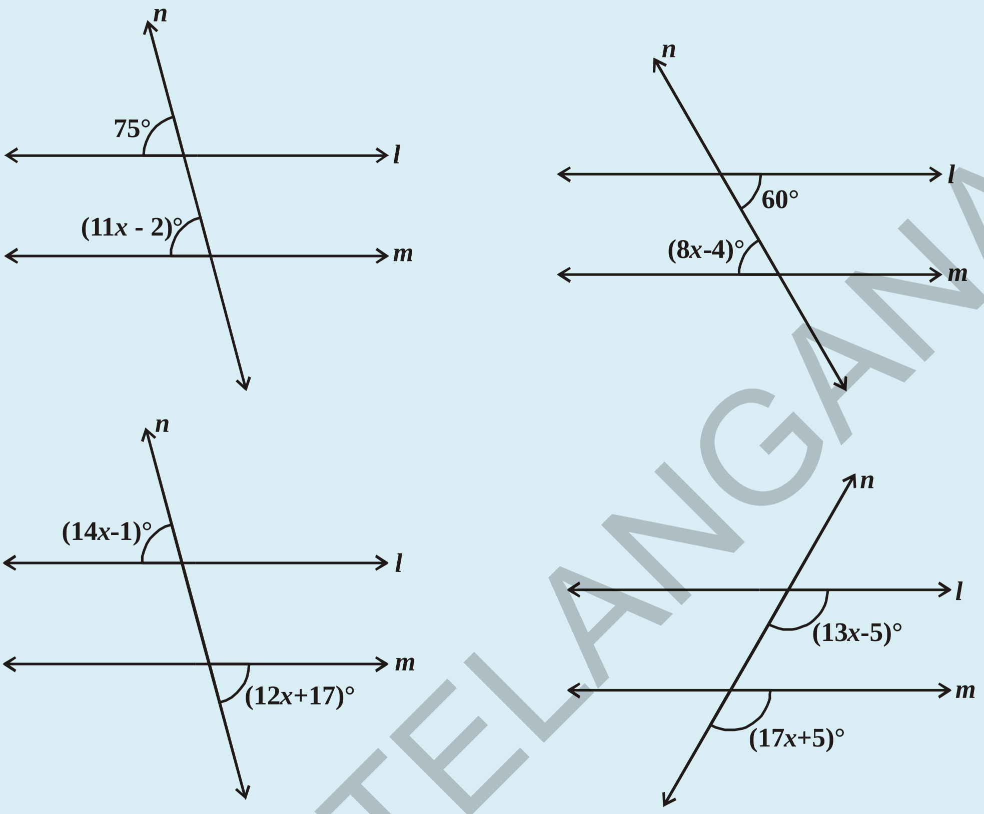






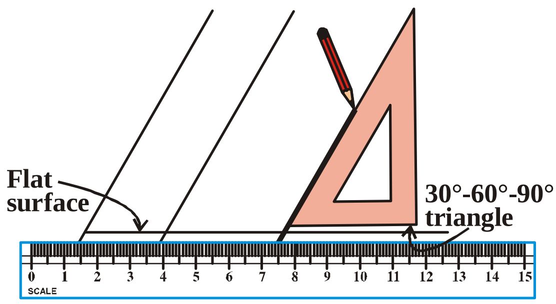


2. If $l \parallel m$, then find 'x' and give reasons.



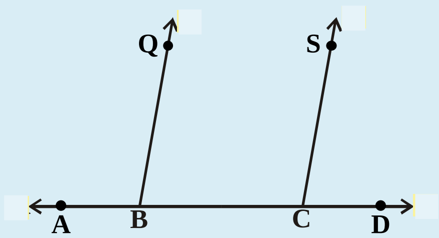
Activity

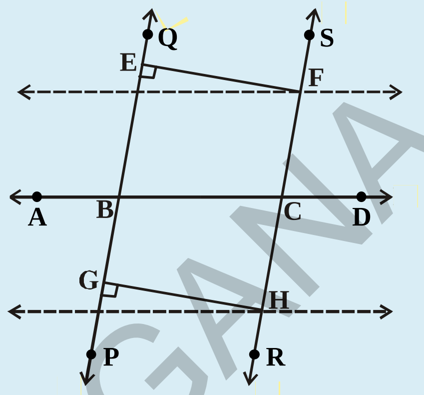
Take a scale and a 'set square'. Arrange the set square on the scale as shown in figure. Along the slant edge of set square draw a line with the pencil. Now slide your set square along its horizontal edge and again draw a line. We observe that the lines are parallel. Why are they parallel? Think and discuss with your friends.



Do This

Draw a line $\overleftrightarrow{AD}$ and mark points B and C on it. At B and C, construct $\angle ABQ$ and $\angle BCS$ equal to each other as shown. Produce QB and SC on the other side of AD to form two lines PQ and RS.

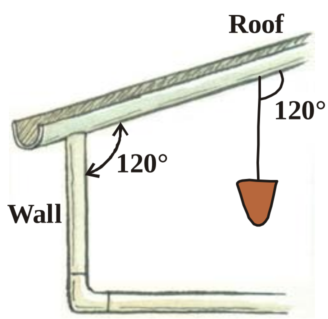




Draw common perpendiculars EF and GH for the two lines PQ and RS. Measure the lengths of EF and GH. What do you observe? What can you conclude from that? Recall that if the perpendicular distance between two lines is the same, then they are parallel lines.

Axiom-1 (Converse of axiom of corresponding angles) : If a transversal intersects two lines such that a pair of corresponding angles are equal, then the two lines are parallel to each other.

A plumb bob has a weight hung at the end of a string and the string here is called a plumb line. The weight pulls the string straight down so that the plumb line is perfectly vertical. Suppose the angle between the wall and the roof is 120^0 and the angle formed by the plumb line and the roof is 120^o. Then the mason concludes that the wall is vertical to the ground. Think, how has he come to this conclusion?



Now, using the converse of the corresponding angles axiom, can we show the two lines are parallel if a pair of alternate interior angles are equal?

In the figure, the transversal $\overleftrightarrow{PS}$ intersects lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ at points Q and R respectively such that the alternate interior angles $\angle RQB$ and $\angle QRC$ are equal.

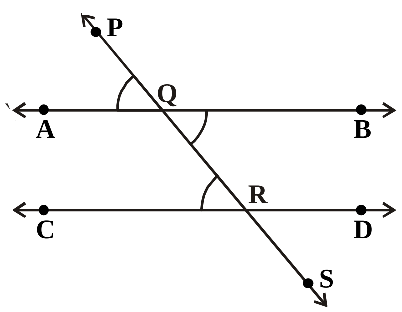
i.e. $\angle RQB = \angle QRC$.

Now we need to prove this AB || CD

$\angle RQB = \angle PQA$ (Why?) ... (1)

But, $\angle RQB = \angle QRC$ (Given) ... (2)

So, from (1) and (2),



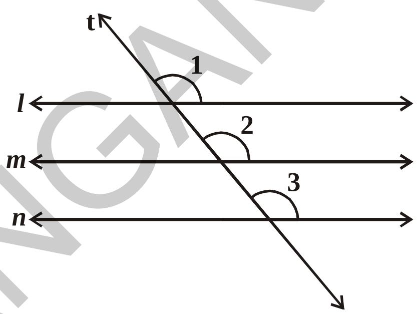
But they are corresponding angles for the pair of lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ with transversal $\overleftrightarrow{PS}$.

So, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ (Converse of corresponding angles axiom)

This result can be stated as a theorem as given below:

Theorem-4.4 : If a transversal intersects two lines such that a pair of alternate interior angles are equal, then the two lines are parallel.

4.4.1 Lines Parallel to the Same Line



If two lines are parallel to the same line, will they be parallel to each other?

Let us check it. Draw three lines *l*, *m* and *n* such that *m* || *l* and *n* || *l*.

Let us draw a transversal 't' on the lines, *l*, *m* and *n*.

Now from the figure $\angle 1 = \angle 2$ and $\angle 1 = \angle 3$ (Corresponding angles axiom)

So, $\angle 2 = \angle 3$ But these two form a pair of corresponding angles for the lines *m* & *n*.

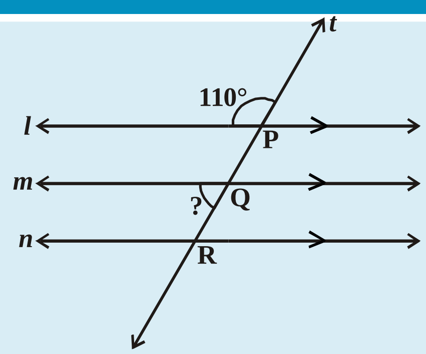
Therefore, you can say that *m* || *n*. (Converse of corresponding angles axiom)

Theorem-4.5 : Lines which are parallel to the same line are parallel to each other.

Try This

(i) Find the measure of the question marked angle in the given figure.

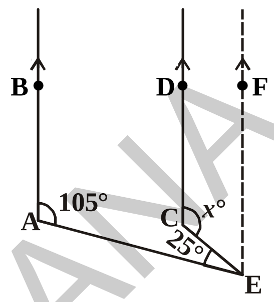
(ii) Find the angles which are equal to $\angle$P.



Now, let us solve some examples related to parallel lines.

Example-8. In the given figure, AB|| CD. Find the value af x.

Solution : From E, draw EF || AB || CD. EF || CD and CE is the transversal.



$\therefore\ \angle ECD + \angle FEC = 180^o$ [$\because$ Co-interior angles]

$\Rightarrow\ x^o + \angle FEC = 180^o \Rightarrow \angle CEF = (180 - x^o)$.

Again, EF || AB and AE is the transversal.

$\angle EAB + \angle FEA = 180^o$ [$\because$ Co-interior angles]

$\Rightarrow 105^o + \angle CEA + \angle FEC = 180^o$

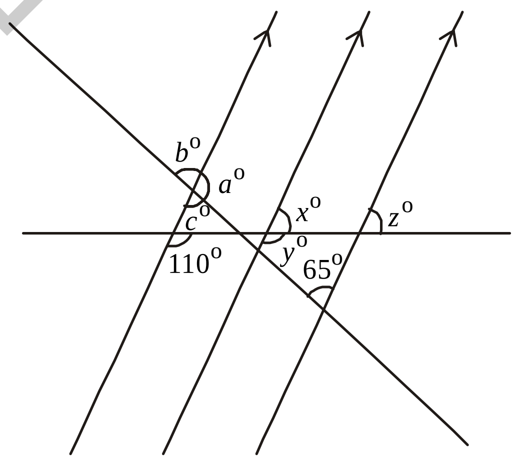
$\Rightarrow\ 105^o + 25^o + (180^o - x^o) = 180^o$

$\Rightarrow\ 310 - x° = 180°$

Hence, $x = 130°$.

Example-9. In the adjacent figure, find the value of x, y, z and a, b, c.

Solution : Clearly, we have



$y^o = 110^o$ ($\because$ Corresponding angles)

$\Rightarrow\ x^o + y^o = 180^o$ (Linear pair)

$\Rightarrow\ x^o + 110^o = 180^o$

$\Rightarrow\ x^o = (180^o - 110^o) = 70^o$.

$z^o = x^o = 70^o$ ($\because$ Corresponding angles)

$c^o = 65^o$ (How?)

$a^o + c^o = 180^o$ [Linear pair]

$\Rightarrow\ a^o + 65^o = 180^o$

$\Rightarrow\ a^o = (180^o - 65^o) = 115^o$.

$b^o = c^o = 65^o$. [$\because$ Vertically opposite angles]

Hence, $a = 115°$, $b = 65°$, $c = 65°$, $x = 70°$, $y = 110°$, $z = 70°$.

Example 10. In the given figure, EF || GH and AB || CD. Find the value of x.

Solution : $4x° = \angle APR$ (Why?)

$\angle APR = \angle PQS$ (Why?)

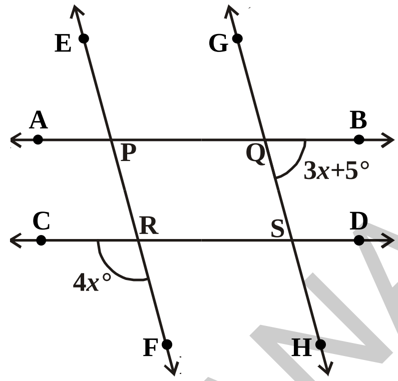
$\angle PQS + \angle SQB = 180°$ (Why?)

$4x° + (3x + 5)° = 180°$

$7x° + 5° = 180°$

$x° = \dfrac{180° - 5°}{7}$

$= 25°$

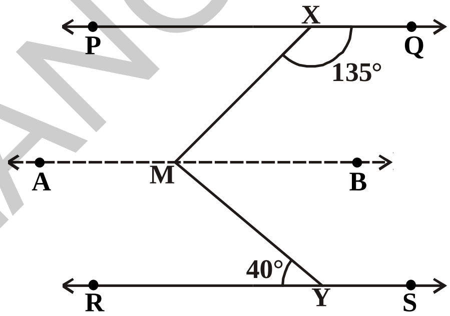


Example-11. In the given figure PQ || RS, $\angle MXQ = 135^o$ and $\angle MYR = 40^o$, find $\angle XMY$.

Solution : Construct a line AB parallel to PQ, through the point M.

Now, AB || PQ and PQ || RS.

$\therefore$ AB || RS



Now, $\angle MXQ + \angle BMX = 180^o$

($\because$ AB || PQ, Interior angles on the same side of the transversal XM)

So, $135° + \angle BMX = 180°$

$\therefore \angle BMX = 45°$ ...(1)

Now, $\angle YMB = \angle MYR$ (Alternate interior angles as AB || RS)

$\therefore \angle YMB = 40°$ ...(2)

Adding (1) and (2), you get

$\angle BMX + \angle YMB = 45° + 40°$

That is, $\angle YMX = 85°$

Example-12. If a transversal intersects two lines such that the bisectors of a pair of corresponding angles are parallel, then prove that the two lines are parallel.

Solution : In the given Figure a transversal $\overleftrightarrow{AD}$ intersects two lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ at two points B and C respectively. Ray $\overrightarrow{BE}$ is the bisector of $\angle QBA$ and ray $\overrightarrow{CF}$ is the bisector of $\angle SCB$; and BE || CF.

We have to prove that PQ || RS. It is enough to prove any one of the following pair:

i. Corresponding angles are equal.

ii. Pair of interior or exterior angles are equal.

iii. Interior angles on same side of the transversal are supplementary.

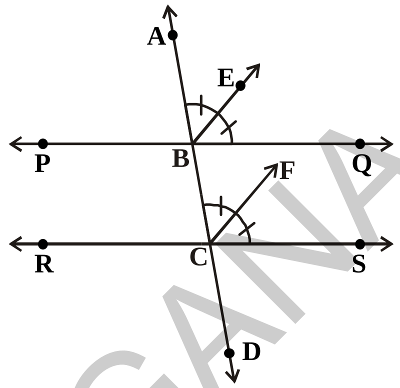
From the figure, we try to prove the pairs of corresponding angles to be equal.

Since, it is given that ray BE is the bisector of $\angle$QBA.

$\angle EBA = \frac{1}{2}\angle QBA$. ... (1)

Similarly, ray CF is the bisector of $\angle$SCB.

$\therefore \angle FCB = \frac{1}{2}\angle SCB$ ... (2)



But for the parallel lines BE and CF; $\overleftrightarrow{AD}$ is a transversal.

Therefore, $\angle EBA = \angle FCB$

(Corresponding angles axiom) ... (3)

From the equations (1), (2) and (3), we get

$$\frac{1}{2}\angle QBA = \frac{1}{2}\angle SCB$$

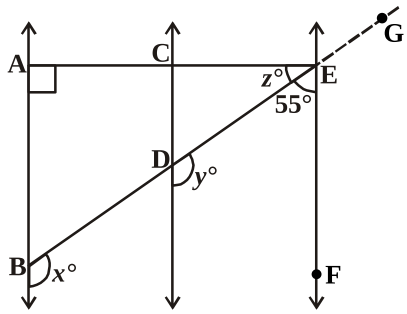
$$\therefore \quad \angle QBA = \angle SCB$$

But, these are the corresponding angles made by the transversal $\overleftrightarrow{AD}$ with lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$; and are equal.

Therefore, PQ || RS (Converse of corresponding angles axiom)

Example-13. In the given figure AB || CD and CD || EF. Also EA $\perp$ AB. If $\angle$BEF = 55°, find the values of x, y and z.

Solution : Extend BE to G.



Now $\angle FEG = 180° - 55°$ (Why?)

$= 125°$

Also $\angle FEG = x = y = 125°$ (Why?)

Now $z = 90° - 55°$ (Why?)

$= 35°$

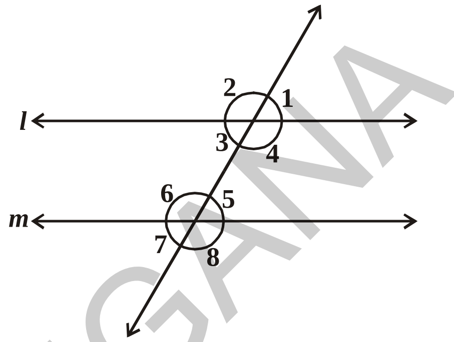
Different ways to prove that two lines are parallel.

1. Showing a pair of corresponding angles are equal.
2. Showing a pair of alternate interior angles are equal.
3. Showing a pair of interior angles on the same side of the transversal are supplementary.
4. In a plane, showing both lines are perpendicular to the same line.
5. Showing both lines are parallel to a third line.

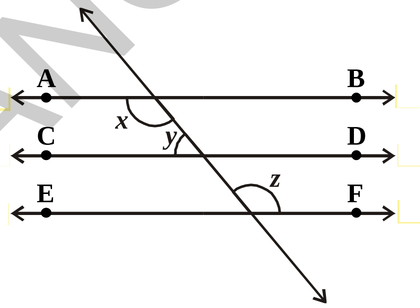
EXERCISE - 4.3

1. It is given that $l \parallel m$ to prove $\angle 1$ is supplement to $\angle 8$. Write reasons for the statement.

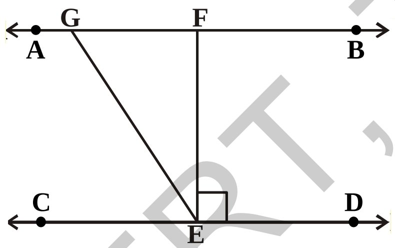
	Statement	Reasons
i.	$l \parallel m$	______________
ii.	$\angle 1 = \angle 5$	______________
iii.	$\angle 5 + \angle 8 = 180°$	______________
iv.	$\angle 1 + \angle 8 = 180°$	______________
v.	$\angle 1$ is supplement to $\angle 8$	______________



2. In the adjacent figure AB || CD; CD || EF and $y : z = 3 : 7$, find x.

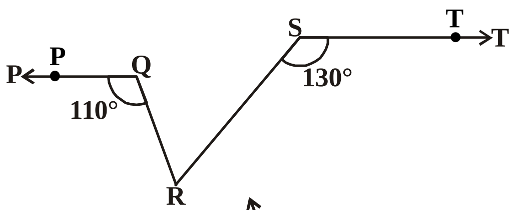


3. In the adjacent figure AB || CD, EF $\perp$ CD and $\angle$DEG = 126°, find $\angle$AGE, $\angle$FEG and $\angle$EGF.

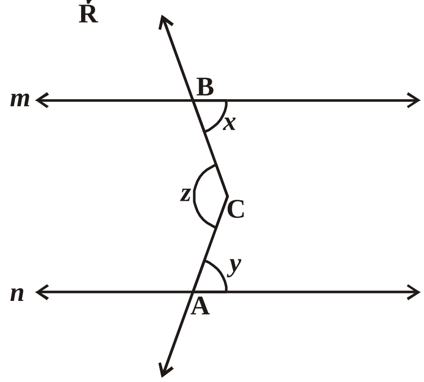


4. In the adjacent figure PQ || ST, $\angle$PQR = 110° and $\angle$RST = 130°, find $\angle$SRQ.

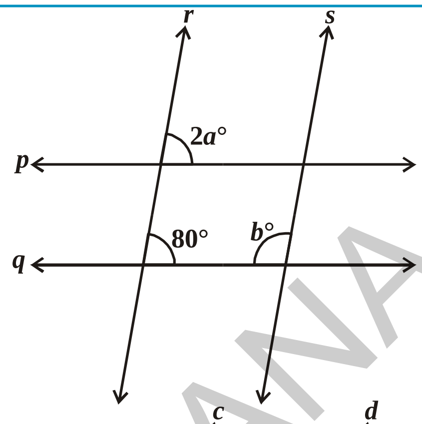
[**Hint :** Draw a line parallel to ST through point R.]



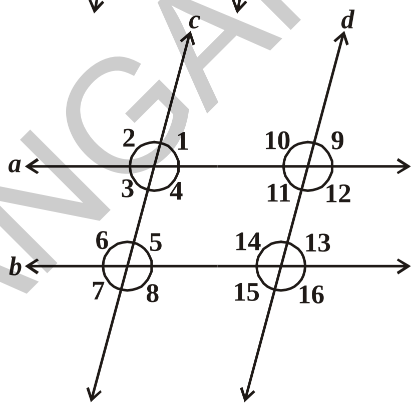
5. In the adjacent figure $m \parallel n$. A, B are any two points on m and n respectively. Let 'C' be an interior, point between the lines m and n. Find $\angle$ACB.

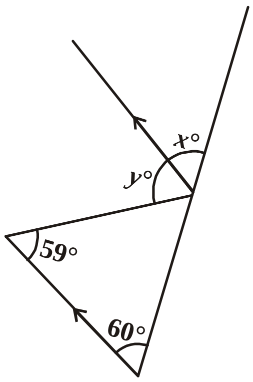


6. Find the value of *a* and *b*, given that *p* || q and *r* || *s*.



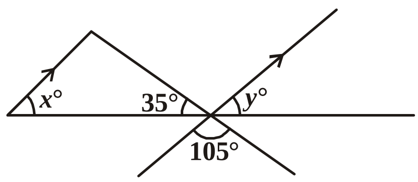
7. If in the figure *a* || *b* and *c* || *d*, then name the angles that are congruent to (i) $\angle 1$ (ii) $\angle 2$.

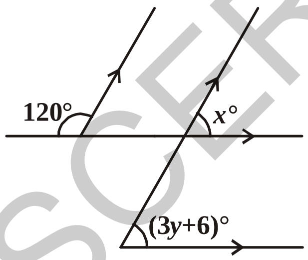




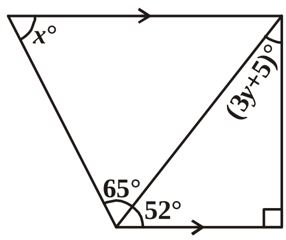
8. In the figure the arrow head segments are parallel. find the value of *x* and *y*.

9. In the adjacent figure the arrow head segments are parallel then find the value of *x* and *y*.





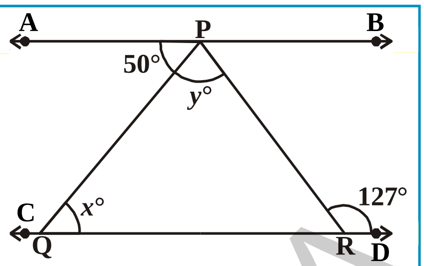
10. Find the value of *x* and *y* from the figure.



11. From the figure find *x* and *y*.

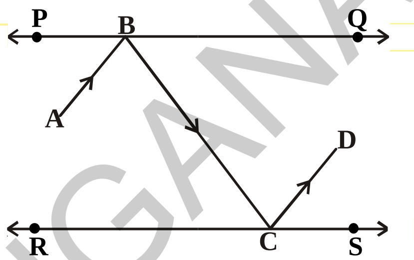
12. Draw figures for the following statement.
"If the two arms of one angle are respectively perpendicular to the two arms of another angle then the two angles are either equal or supplementary".

13. In the given figure, if AB || CD, $\angle APQ = 50°$ and $\angle PRD = 127°$, find x and y.

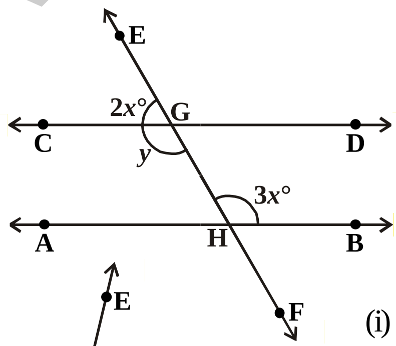


14. In the adjacent figure PQ and RS are two mirrors placed parallel to each other. An incident ray $\overrightarrow{AB}$ strikes the mirror PQ at B, the reflected ray moves along the path $\overrightarrow{BC}$ and strikes the mirror RS at C and again reflected back along $\overrightarrow{CD}$. Prove that AB || CD.

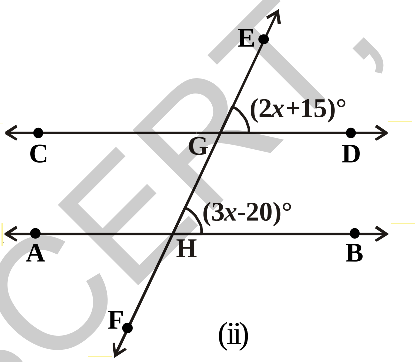
[Hint : Perpendiculars drawn to parallel lines are also parallel.]



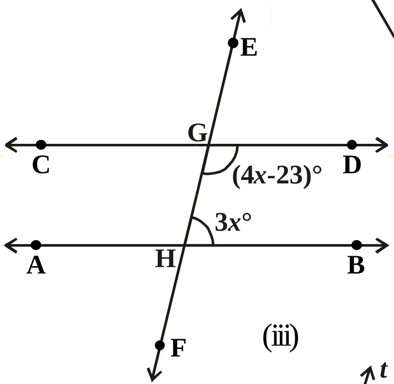
15. In the figures given below AB || CD. EF is the transversal intersecting AB and CD at G and H respectively. Find the values of x and y. Give reasons



(i)



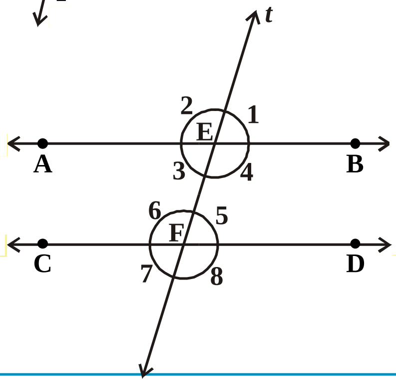
(ii)



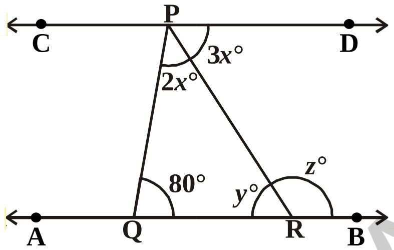
(iii)

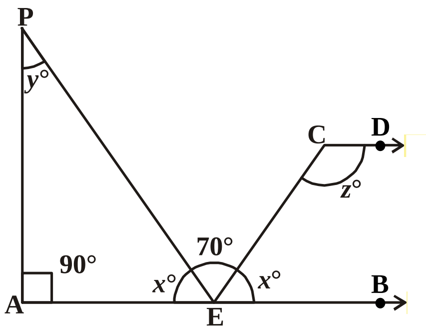
16. In the adjacent figure, AB || CD, 't' is a transversal intersecting E and F respectively.

If $\angle 2 : \angle 1 = 5 : 4$, find the measure of each marked angles.



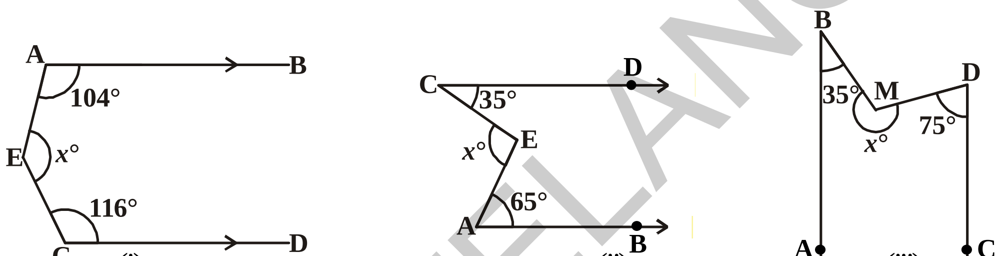
17. In the adjacent figure AB || CD. Find the value of angles $x°$, $y°$ and $z°$.





18. In the adjacent figure AB || CD. Find the values of x, y and z angles.

19. In each of the following figures AB || CD. Find the value of x in each case.



(i) (ii) (iii)

4.5 Angle Sum Property of a Triangle

Let us now prove that the sum of the interior angles of a triangle is 180°.

Activity

- Draw and cut out a large triangle from sheet as shown in the figure (i).
- Number the angles and tear them off shown in fig (i).
- Place the three angles adjacent to each other to form one angle. as shown in figure (ii).

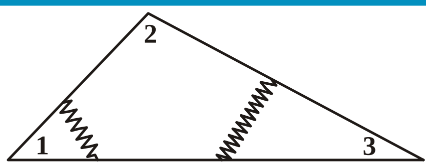


fig. (i)

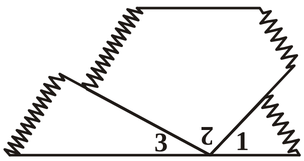
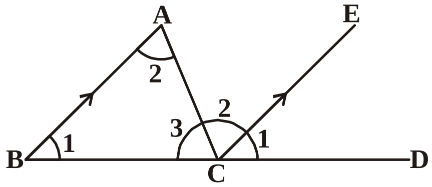


fig. (ii)

1. Identify angle formed by the three adjacent angles? What is its measure?

2. Write about the sum of the measures of the angles of a triangle.

Theorem-4.6 : The sum of the angles of a triangle is 180°.



Given : ABC is a triangle.

R.T.P. : $\angle A + \angle B + \angle C = 180^0$

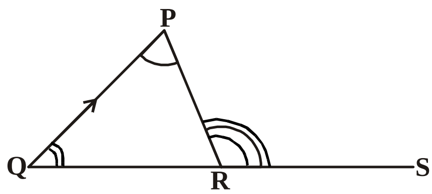
Construction : Produce BC to a point D and through 'C' draw a line CE parallel to BA

Proof :

BA‖CE	[By construction]
$\angle CBA = \angle DCE$ .....(1)	[By corresponding angles axiom.]
$\angle BAC = \angle ECA$ .....(2)	[Alternate interior angles for the parallel lines AB and CE]
$\angle ACB = \angle ACB$ .....(3)	[Same angle]
$\angle CBA + \angle BAC + \angle ACB =$ $\angle DCE + \angle ECA + \angle ACB$	[Adding the above three equations]
But $\angle DCE + \angle ECA + \angle ACB = 180^0$	[Sum of angles at a point on a straight line]
$\therefore \angle CBA + \angle BAC + \angle ACB = 180^0$	
$\angle A + \angle B + \angle C = 180^0$	

You know that when a side of a triangle is produced there forms an exterior angle of the triangle

When side QR is produced to point S, $\angle SRP$ is called an exterior angle of ΔPQR.



Is $\angle PRQ + \angle SRP = 180°$? (Why?) .....(1)

Also, see that

$\angle PRQ + \angle RQP + \angle QPR = 180°$ (Why?) .....(2)

From (1) and (2), we can see that $\angle PRQ + \angle SRP = \angle PRQ + \angle RQP + \angle QPR$

$\therefore \angle SRP = \angle RQP + \angle QPR$

This result can be stated in the form of a theorem as given below

Theorem-4.7 : If a side of a triangle is produced, then the exterior angle so formed is equal to the sum of the two interior opposite angles.

It is obvious from the above theorem that an exterior angle of a triangle is always greater than either of its interior opposite angles.

Now, let us solve some examples based on the above.

THINK, DISCUSS AND WRITE

If the sides of a triangle are produced in order, what will be the sum of exterior angles thus formed?

Example-14. The angles of a triangle are $(2x)°$, $(3x+5)°$ and $(4x - 14)°$.

Find the value of x and the measure of each angle of the triangle.

Solution : We know that the sum of the angles of a triangle is 180^o.

$\therefore\ 2x° + 3x° + 5° + 4x° - 14° = 180° \Rightarrow 9x° - 9° = 180°$

$\Rightarrow 9x° = 180° + 9° = 189°$

$\Rightarrow x° = \dfrac{189°}{9°} = 21.$

$\therefore\ 2x° = (2\times 21)° = 42°,$

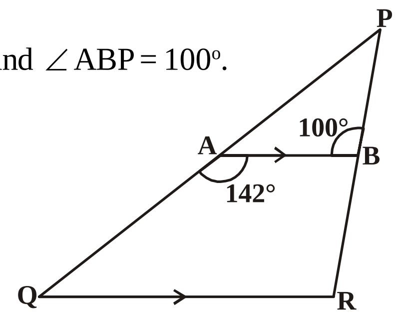
$(3x+5)° = [(3\times 21+5)]° = 68°.$

$(4x - 14)° = [(4\times 21) - 14]° = 70°$

Hence, the angles of the triangle are 42^o, 68^o and 70^o.

Example-15. In the adjacent figure, AB || QR, $\angle$BAQ = 142^o and $\angle$ABP = 100^o.

Find (i) $\angle$APB (ii) $\angle$AQR and (iii) $\angle$QRP,



Solution : (i) Let $\angle$APB = x^o,

Side PA of ΔPAB is produced to Q.

$\therefore$ Exterior angle $\angle$QAB = $\angle$PBA + $\angle$APB

$\Rightarrow 142^o = 100^o + x^o$

$\Rightarrow x^o = (142^o - 100^o) = 42^o.$

$\therefore\ \angle APB = 42^o,$

(ii) Now, AB || QR and PQ is a transversal.

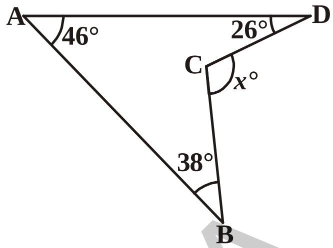
$\therefore\ \angle QAB + \angle RQA = 180^0$ [Sum of co-interior angles is 180^o]

$\Rightarrow\ 142^o + \angle RQA = 180^o,$

$\therefore\ \angle RQA = (180^o - 142^o) = 38^o.$

(iii) Since AB || QR and PR is a transversal.

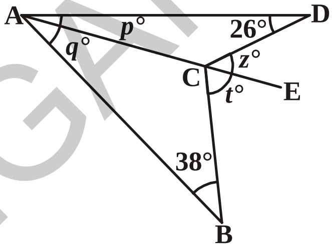
Example-16. Using information given in the adjacent figure, find the value of x.



Solution : In the given figure, ABCD is a quadrilateral. Let us try to make it as two triangles.

Join AC and produce it to E.

Let $\angle$DAE = p°, $\angle$BAE = q°, $\angle$ DCE = z° and $\angle$ECB = t°. Since the exterior angle of a triangle is equal to the sum of the interior opposite angles, we have :



$z^\circ = p^\circ + 26^\circ$

$t^\circ = q^\circ + 38^\circ$

$\therefore\ z^\circ + t^\circ = p^\circ + q^\circ + (26 + 38)^\circ = p^\circ + q^\circ + 64^\circ$

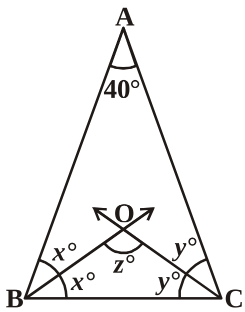
But, $p^\circ + q^\circ = 46$. $\quad (\because \angle DAB = 46^o)$

So, $z^\circ + t^\circ = 46 + 64 = 110^\circ$.

Hence $x^\circ = z^\circ + t^\circ = 110^\circ$.

Example-17. In the given figure $\angle$A = 40°. If $\overrightarrow{BO}$ and $\overrightarrow{CO}$ are the bisectors of $\angle$B and $\angle$C respectively. Find the measure of $\angle$ BOC.

Solution : We know that BO is the bisector of $\angle$ B and CO is the bisector of $\angle$C.



Let $\angle$CBO = $\angle$OBA = x° and $\angle$OCB = $\angle$ACO = y°.

Then, $\angle$ B = (2x)°, $\angle$ C = (2y)° and $\angle$ A = 40°.

But, $\angle$A + $\angle$B + $\angle$ C = 180°. (How?)

$2x° + 2y° + 40° = 180°$

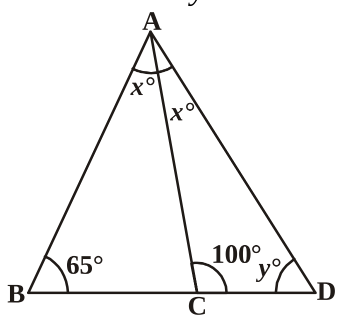
$\Rightarrow\ 2(x + y)° = 140°$

$= x° + y° = \dfrac{140°}{2} = 70°.$

Hence, $\angle$BOC = 180° - 70° = 110°.

Example-18. Using information given in the adjacent figure, find the values of x and y.

Solution : Side BC of ΔABC has been produced to D.



Exterior $\angle$DCA = $\angle$CBA + $\angle$BAC

$\therefore\ 100° = 65^o + x^o$

$\Rightarrow\ x^o = (100^o - 65^o) = 35^o.$

$\therefore\ \angle CAD = \angle BAC = 35^o$

In ΔACD, we have :

$\angle CAD + \angle DCA + \angle ADC = 180^{o}$ (Angle sum property of triangle)

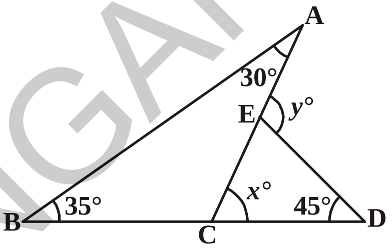
$\Rightarrow 35^{o} + 100^{o} + y^{o} = 180^{o}$

$\Rightarrow 135^{o} + y^{o} = 180^{o}$

$\Rightarrow y^{o} = (180^{o} - 135^{o}) = 45^{o}$

Hence, $x = 35°, y = 45°$.

Example-19. Using information given in the adjacent figure, find the value of x and y.



Solution : Side BC of ΔABC has been produced to D.

$\therefore$ Exterior angle $\angle ACD = \angle BAC + \angle CBA$

$\Rightarrow \quad x^{o} = 30^{o} + 35^{o} = 65^{o}$.

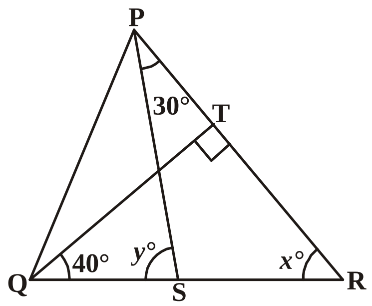
Again, side CE of ΔDCE has produced to A.

$\therefore$ Exterior angle $\angle DEA = \angle EDC + \angle ECD$

$\Rightarrow \quad y = 45 + x^{o} = 45^{o} + 65^{o} = 110^{o}$.

Hence, $x = 65°$ and $y = 110°$.

Example-20. In the adjacent fig. if $QT \perp PR$, $\angle RQT = 40°$ and $\angle SPR = 30°$, find values of x and y.



Solution : In ΔRQT,

$90° + 40° + x° = 180°$ (Angle sum property of a triangle)

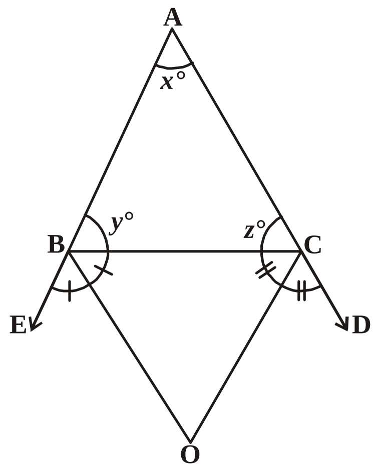
$\therefore \quad x° = 50°$

Now, $\quad y° = \angle SPR + x°$ (Exterior angle of traingle)

$\therefore \quad y° = 30° + 50°$

$= 80°$

Example-21. In the adjacent figure the sides AB and AC of ΔABC are produced to points E and D respectively. If bisectors BO and CO of $\angle CBE$ and $\angle BCD$ respectively meet at point O, then prove that $\angle COB = 90° - \frac{1}{2}\angle BAC$.



Solution : Ray BO is the bisector of $\angle EBC$.

$\therefore \quad \angle OBC = \frac{1}{2}\angle EBC$

$= \frac{1}{2}(180° - y°)$

Similarly, ray CO is the bisector of $\angle$BCD.

$$\therefore \quad \angle BCO = \frac{1}{2}\angle BCD$$

$$= \frac{1}{2}(180° - z°)$$

$$= 90° - \frac{z°}{2} \qquad \text{...(2)}$$

In ΔBOC, $\angle COB + \angle BCO + \angle OBC = 180°$ ...(3)

Substituting (1) and (2) in (3), you get

$$\angle COB + 90° - \frac{z°}{2} + 90° - \frac{y°}{2} = 180°$$

So, $$\angle COB = \frac{z°}{2} + \frac{y°}{2}$$

or, $$\angle COB = \frac{1}{2}(y° + z°) \qquad \text{... (4)}$$

But, $x° + y° + z° = 180°$ (Angle sum property of a triangle)

$\Rightarrow$ $y° + z° = 180° - x°$

$\therefore$ Equation (4) becomes

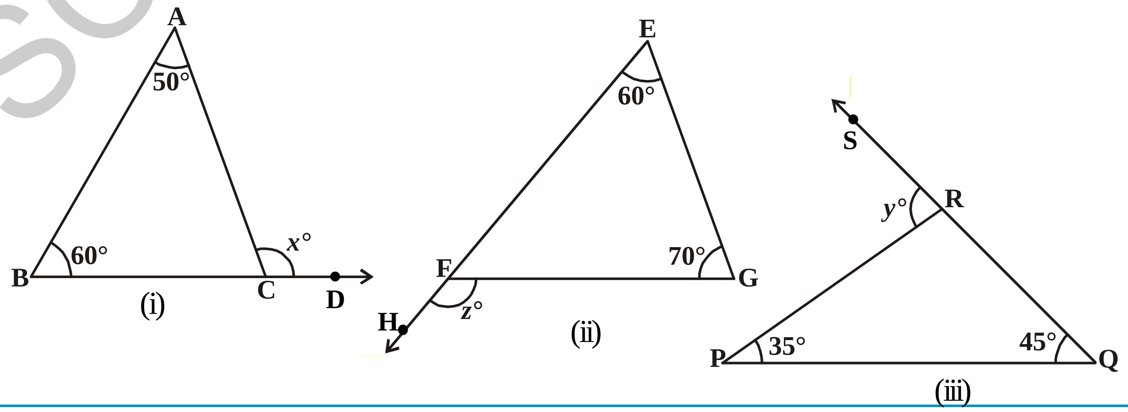
$$\angle COB = \frac{1}{2}(180° - x°)$$

$$= 90° - \frac{x°}{2}$$

$$= 90° - \frac{1}{2}\angle BAC$$

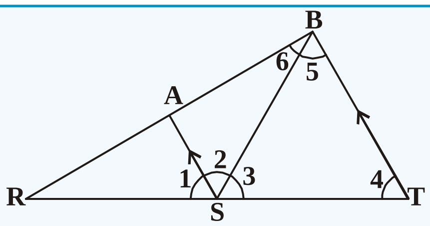
Exercise 4.4

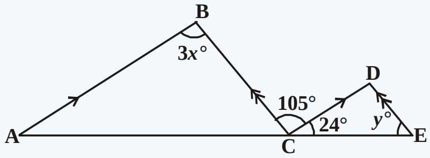
1. In the given triangles, find out x, y and z.



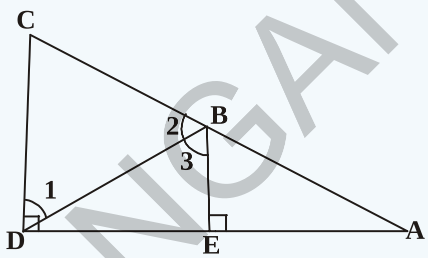
2. In the given figure AS || BT; $\angle 4 = \angle 5$

$\overrightarrow{SB}$ bisects $\angle$TSA. Find the measure of $\angle 1$

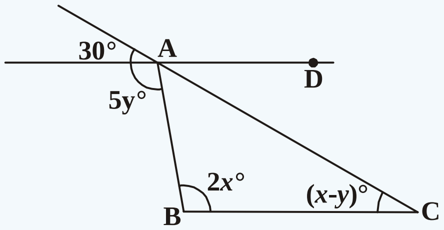




3. In the given figure AB || CD; BC || DE then find the values of x and y.

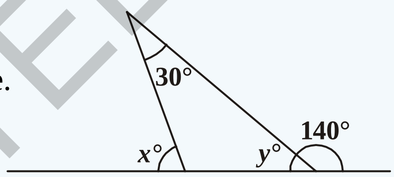


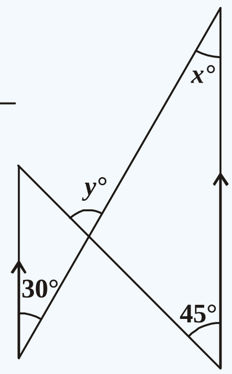
4. In the adjacent figure BE $\perp$ DA and CD$\perp$DA then prove that $\angle 1 \cong \angle 3$.



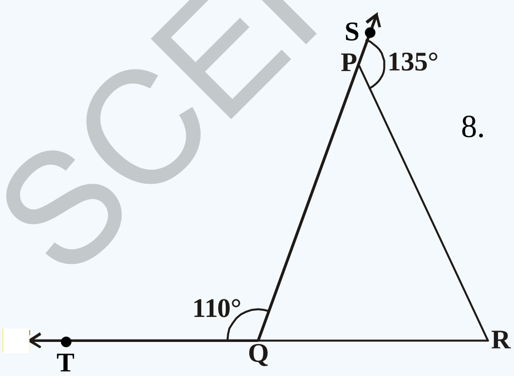
5. Find the values of x, y for which the lines AD and BC become parallel.

6. Find the values of x and y in the figure.

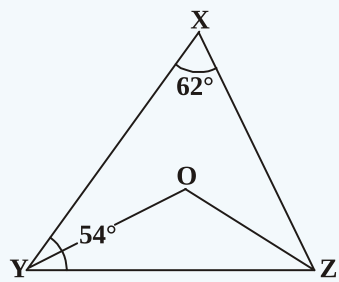




7. In the given figure segments shown by arrow heads are parallel. Find the values of x and y.

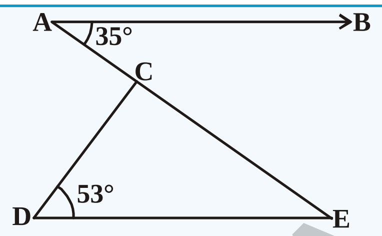


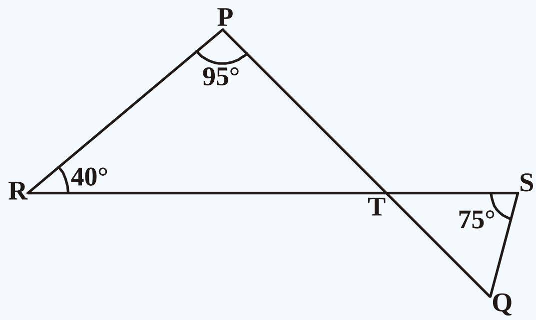
8. In the given figure sides QP and RQ of ΔPQR are produced to points S and T respectively. If $\angle$RPS = 135° and $\angle$PQT = 110°, find $\angle$PRQ.



9. In the given figure, $\angle X = 62°$, $\angle ZYX = 54°$. In ΔXYZ If YO and ZO are the bisectors of $\angle$XYZ and $\angle$XZY respectively find $\angle$OZY and $\angle$YOZ.

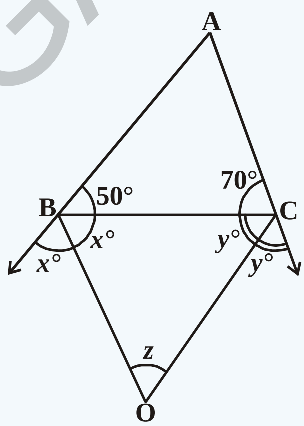
10. In the given figure if AB || DE, $\angle$BAC = 35° and $\angle$CDE = 53°, find $\angle$DCE.

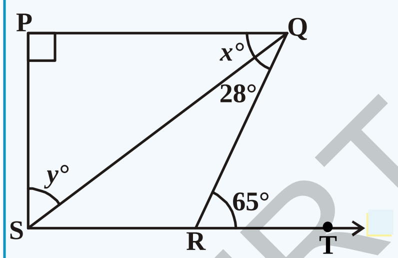




11. In the given figure if line segments PQ and RS intersect at point T, such that $\angle$TRP = 40°, $\angle$RPT = 95° and $\angle$TSQ = 75°, find $\angle$SQT.

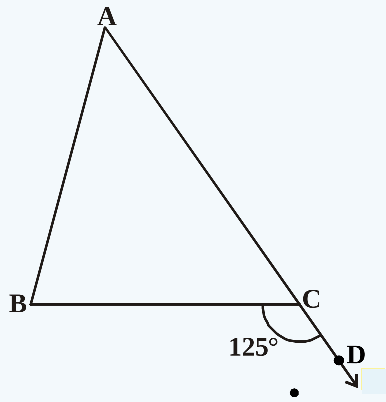
12. In the adjacent figure, ABC is a triangle in which $\angle$B = 50° and $\angle$C = 70°. Sides AB and AC are produced. If 'z' is the measure of the angle between the bisectors of the exterior angles so formed, then find 'z'.

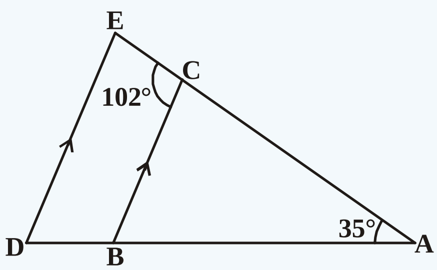




13. In the given figure if PQ $\perp$ PS, PQ || SR, $\angle$SQR = 28° and $\angle$TRQ = 65°, then find the values of x and y.

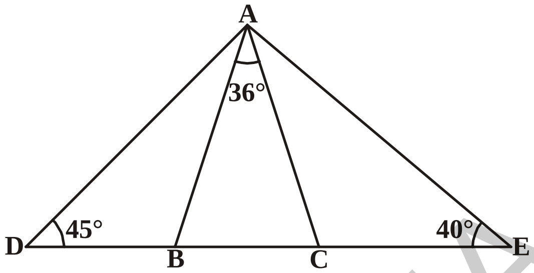
14. In the given figure ΔABC side AC has been produced to D. $\angle$BCD = 125° and $\angle$A : $\angle$B = 2 : 3, find the measure of $\angle$A and $\angle$B.

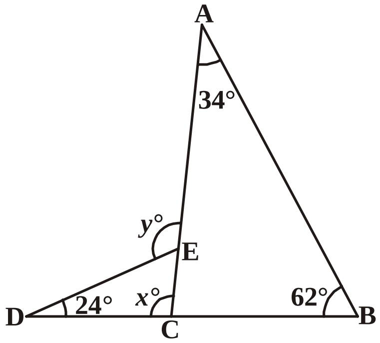




15. In the adjacent figure, it is given that, BC || DE, $\angle$BAC = 35° and $\angle$BCE = 102°. Find the measure of (i) $\angle$BCA (ii) $\angle$ADE and (iii) $\angle$CED.

16. In the adjacent figure, it is given that AB = AC, $\angle BAC = 36^{o}$, $\angle BDA = 45^{o}$ and $\angle AEC = 40^{o}$. Find (i) $\angle ABC$ (ii) $\angle ACB$ (iii) $\angle DAB$ (iv) $\angle EAC$.





17. Using information given in the figure, calculate the value of x and y.

WHAT WE HAVE DISCUSSED ?

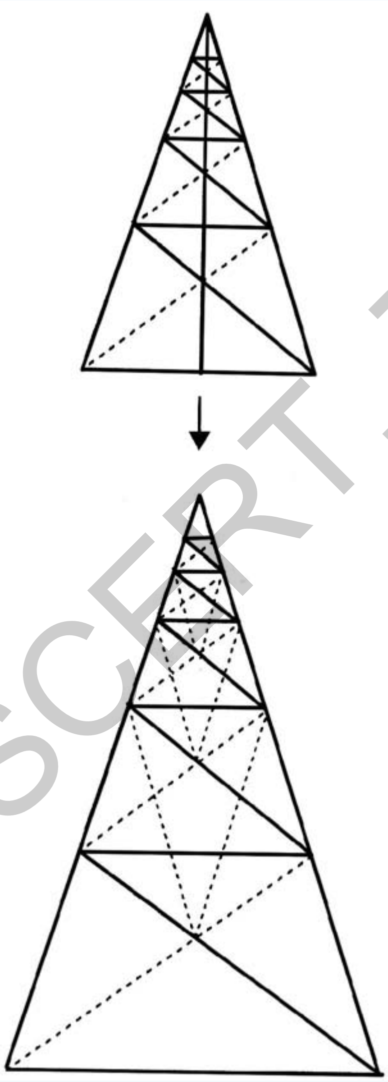
- **Linear pair axiom:** If a ray stands on a straight line, then the sum of the two adjacent angles so formed is 180°.
- **Converse of linear pair axiom:**
  If the sum of two adjacent angles is 180°, then the non-common arms of the angles form a line.
- **Theorem**: If two lines intersect each other, then the vertically opposite angles are equal.
- **Axiom of corresponding angles:** If a transversal intersects two parallel lines, then each pair of corresponding angles are equal.
- **Theorem**: If a transversal intersects two parallel lines, then each pair of alternate interior angles are equal.
- **Theorem**: If a transversal intersects two parallel lines, then each pair of interior angles on the same side of the transversal are supplementary.
- **Converse of axiom of corresponding angles**:
  If a transversal intersects two lines such that a pair of corresponding angles are equal, then the two lines are parallel to each other.
- **Theorem**: If a transversal intersects two lines such that a pair of alternate interior angles are equal, then the two lines are parallel.

K1M4Y5

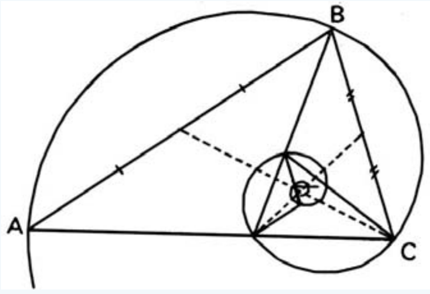
- **Theorem**: If a transversal intersects two lines such that a pair of interior angles on the same side of the transversal are supplementary, then the two lines are parallel.
- **Theorem**: Lines which are parallel to a given line are parallel to each other.
- **Theorem:** The sum of the angles of a triangle is 180º.
- **Theorem**: If a side of a triangle is produced, then the exterior angle so formed is equal to the sum of the two interior opposite angles.

Do You Know?

The Self-generating Golden Triangle



The golden triangle is an isosceles triangle with base angles 72^o and the vertex angle 36^o. When both of these base angles are bisected the two new triangles produced are also golden triangles. This process can be continued indefinitely up the legs of the original golden triangle, and an infinite number of golden triangles will appear as if they are unfolding.



As this diagram shows, the golden triangle also produces the equi-angular spiral and the golden ratio, $\phi = |AB| / |BC| = 1.618 ...$

From these infinite climbing golden triangles one can also construct inside them an infinite number of climbing pentagrams. Note the five points of the penta-gram are also golden triangles.

Chapter

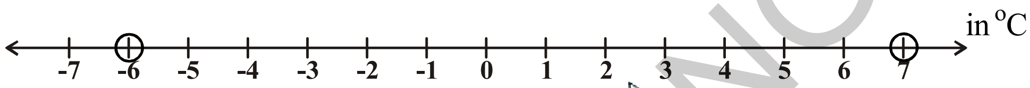
5

Co-Ordinate Geometry

Q8T3Y5

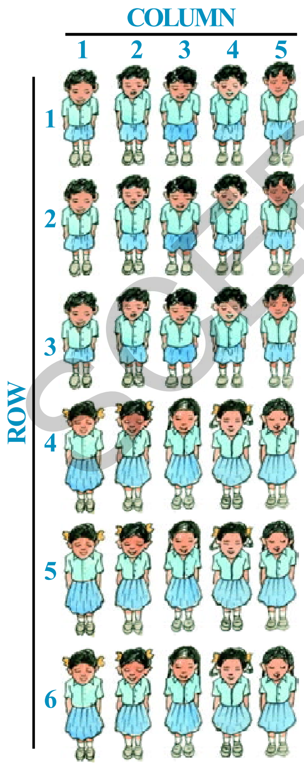
5.1 Introduction

The minimum and the maximum temperatures of Kufri in Himachal Pradesh on a particular day in the month of December were - 6°C and 7°C. Can you represent them on a number line?



Here the numberline acts as a reference scale to indicate the status of temperature on a particular day.

Let us observe the situation as shown in the adjacent picture. Eight persons A, B, C, D, E, F, G and H are standing in a queue. From the ticket counter, A is the first and H is the last person in the queue. With reference to the cafe, 'H' becomes the first and 'A' will become the last person. You might have observed that the positioinal value of the object changes along with the change of reference.



Let us observe another example. In a games period, the students of class IX assembled as shown in the picture. Can you say where Sudha is standing in the picture?

Rama said "Sudha is standing in 2nd column."

Pavani said "Sudha is standing in 4th row."

Nasima said "Sudha is standing in 2nd column and 4th row."

Whom of the above gave correct information? Can you identify Sudha with the information given by Nasima? Can you locate the position of Madhavi (who is standing in 1st column and 5th row?)

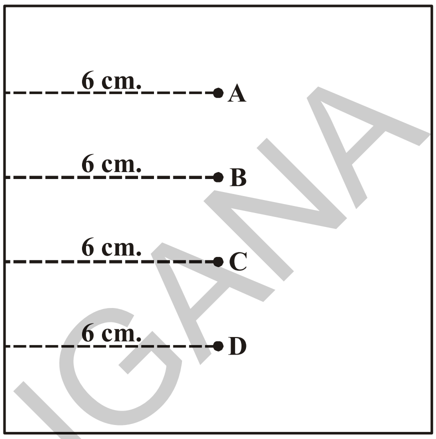
Identify the students who are standing in following positions.

(i) (3rd column, 6th row) by Sita
(ii) (5th column, 2nd row) by Raju

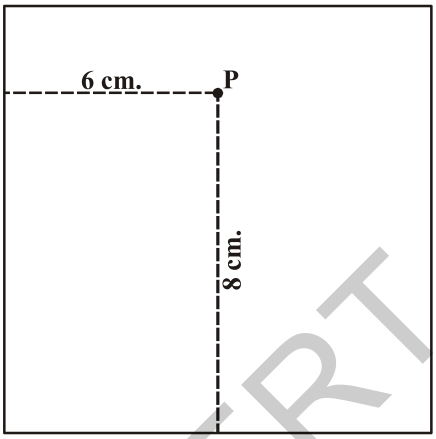
In the above example can you say how many references did you consider? What are they?

Let us discuss one more situation.

A teacher asked her students to mark a point on a sheet of paper. The hint given by the teacher is "the point should be at a distance of 6 cm from the left edge." Some of the students marked the point as shown in the figure.



In the figure which point do you suppose is correct? Since each point A,B,C and D is at a distance of 6 cm from the left edge, no point can be denied. To fix the exact position of the point one more information is needed. To fix its exact position, another reference, say, the distance from the edge of the top or bottom has to be given.



Suppose the teacher says that the point is at a distance of 6 cm from the left edge and at a distance of 8 cm from the bottom edge, now how many points with this description can be marked?

Only a single point can be marked. So, how many references do you need to fix the position of a point?

We need two references to describe for fixing the exact position of a point. The position of the point is denoted by (6,8). If you say "a point is marked at a distance of 7 cm from the top." Can you trace its exact position? Discuss with your friends.

Do This

Describe the seating position of any five students in your classroom.

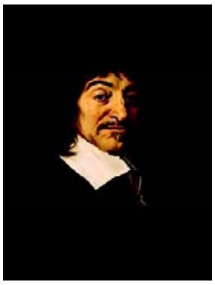
Activity (Ring Game)

Have you seen 'Ring game' in exhibitions? We throw rings on the objects arranged in rows and columns. Observe the following picture.

Complete the following table with suitable numbers

Object	Column	Row	Position
Purse	3	4	(3,4)
Match box	....	3	(,3)
Clip	....	....	....
Teddy	....	....	....
Soap	....	....	....

Is the object in 3rd column and 4th row is same as 4th column and 3rd row?

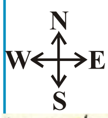


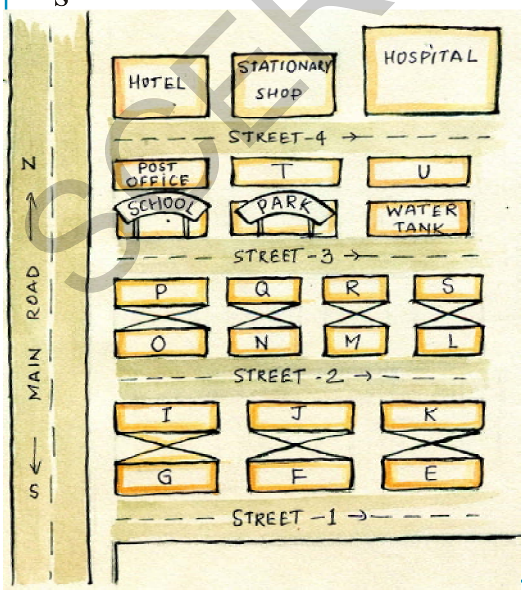
The representation of a point on a plane with idea of two references led to development of new branch of mathematics known as Coordinate Geometry.

Rene Descartes (1596-1650), a French mathematician and philosopher has developed the study of Co-ordinate Geometry. He found an association between algebraic equations and geometric curves and figures. In this chapter we shall discuss about the point and also how to plot the points on a co-ordinate plane.

EXERCISE 5.1

1. In a locality, there is a main road along North-South direction. The map is given below. With the help of the picture answer the following questions.





(i) What is the 3rd object on the left side in street no. 3?

(ii) Find the name of the 2nd house which is in right side of street 2.

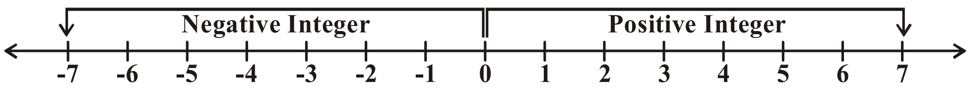
(iii) Locate the position of Mr. K's house.

(iv) Describe the position of the post office.

(v) Describe the location of the hospital.

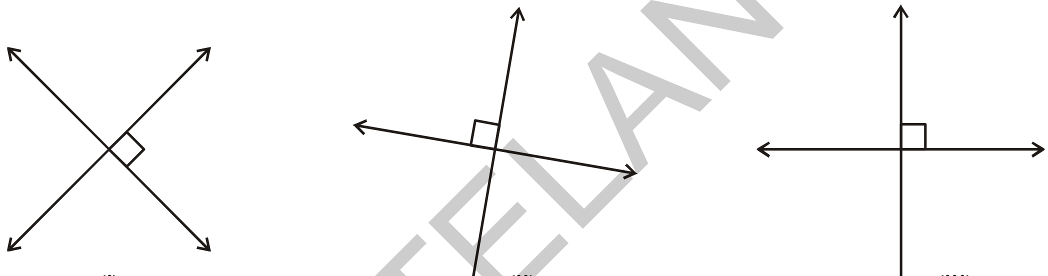
5.2 Cartesian System

We use number line to represent the numbers by marking points on the line. Observe the following integer line.



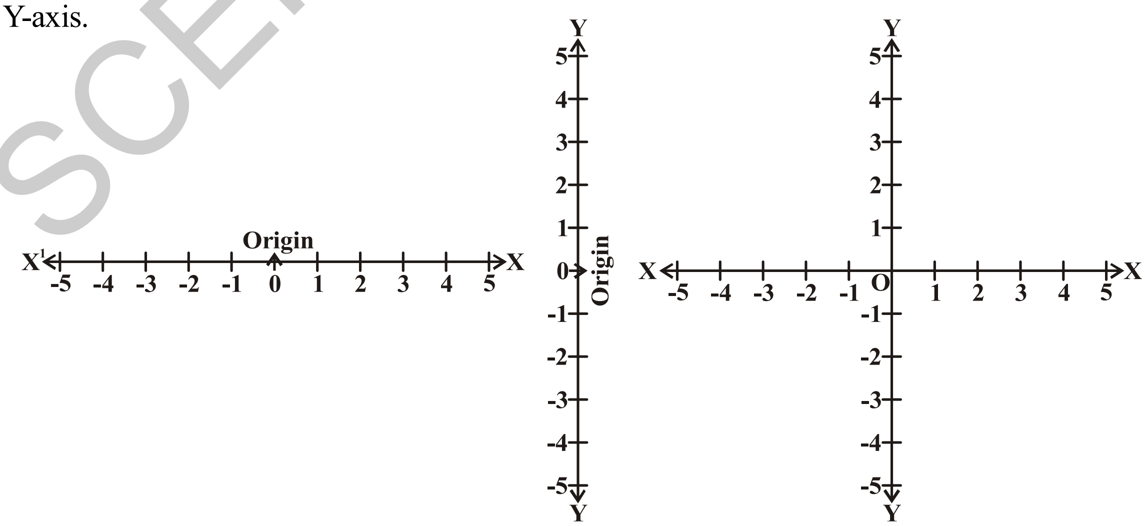
Points marked with equidistance on either side from the fixed point zero. The fixed point corresponding to zero is called orgin, it is denoted by 'O'. All positive integers are shown on the right side of zero and all negative integers on its left side.

We take two number lines, perpendicular to each other in a plane. We locate the position of a point with reference to these two lines. Observe the following figure.

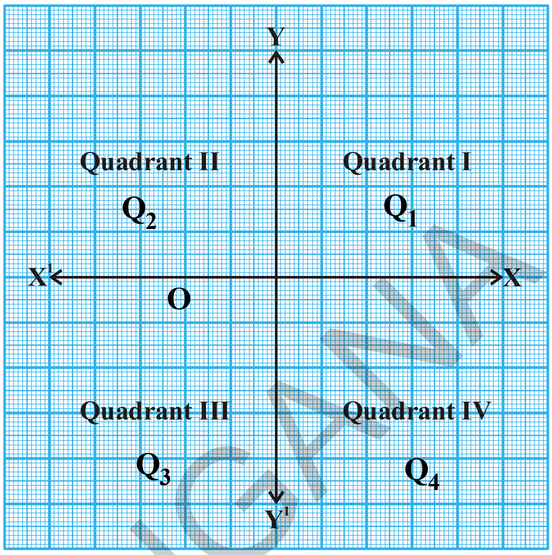


(i) (ii) (iii)

The perpendicular lines may be in any direction as shown in the figures. But, when we choose these two lines to locate a point in a plane in this chapter, for the sake of convenience we take one line horizontally and the other vertically as in fig. (iii). We draw a horizontal number line and a vertical number line intersecting at a point perpendicular to each other. The point of intersection is denoted as origin. The horizontal number line XX^1 is known as X-axis and the vertical number line YY^1 is known as Y-axis.

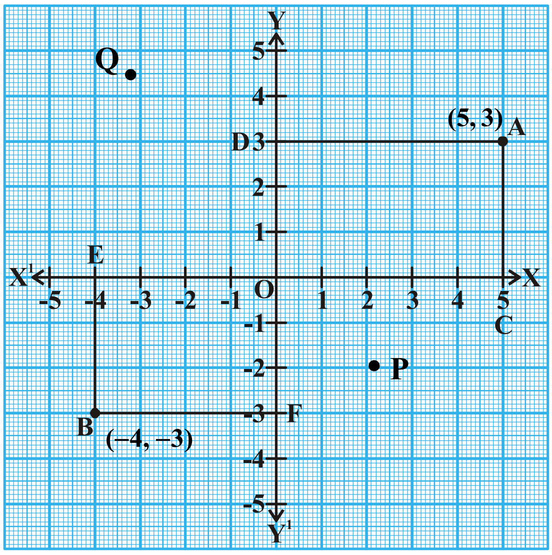


The point where XX^1 and YY^1 cross each other is called the origin, and is denoted by 'O'. Since the positive numbers lie on the directions $\overrightarrow{OX}$, is called the positive direction of the X-axis, similarly $\overrightarrow{OY}$ is the positive Y-axis respectively. Also $\overrightarrow{OX^1}$ and $\overrightarrow{OY^1}$ are called the negative directions of the X-axis and the Y-axis respectively. We can observe that the axes (plural of axis) divide the plane into four parts. These are called first, second, third and fourth quadrants respectively. These four parts are called the quadrants and are denoted by Q_1, Q_2, Q_3 and Q_4 in anti clockwise direction. The plane here is called the cartesian plane (named after Rene Descartes) or co-ordinate plane or XY-plane. The axes are called the coordinate axes.



5.2.1 Locating a Point

Now let us see how to locate a point in the coordinate system. Observe the following graph. Two axes X and Y are drawn on a graph paper. A and B are any two points on it. Can you say the quadrants to which the points A and B belong to?



The point A is in the first quadrant (Q_1) and the point B is in the third quadrant (Q_3). Now let us see the distances of A and B from the axes. For this we draw the perpendiculars AC on the X-axis and AD on the Y-axis. Similarly, we draw perpendiculars BE and BF as shown in figure.

We can observe the following

(i) The perpendicular distance of the point A from the Y-axis measured along the positive direction of X-axis is AD=OC= 5 units. We call this as X-coordinate of 'A'.

(ii) The perpendicular distance of the point A from the X-axis measured along the positive direction of the Y-axis is AC=OD=3 units. We call this as Y-coordinate of 'A'. Therefore coordinates of 'A' are (5, 3)

(iii) The perpendicular distance of the point B from the Y-axis measured along the negative direction of X-axis is OE=BF= 4 units. i.e. at –4 on X-axis. We call this as X-coordinate of 'B'.

(iv) The perpendicular distance of the point B from the X-axis measured along the negative direction of Y-axis is OF = EB = 3 units. i.e. at –3 on Y-axis. We call this as Y-coordinate of 'B' and (–4, –3) are coordinates of 'B'.

Now using these distances, how can we locate the point? We write the coordinates of a point in the following method.

(i) The x-coordinate of a point is the distance from origin to foot of perpendicular on X-axis. The x-coordinate is also called the abscissa.

The x-coordinate (abscissa) of P is 2.

The x-coordinate (abscissa) of Q is –3.

(ii) The y-coordinate of a point is the distance from origin to foot of perpendicular on Y-axis.

The y-coordinate is also called the ordinate.

The y-coordinate or ordinate of P is –2.

The y-coordinate or ordinate of Q is 4.

Hence the coordinates of P are (2, –2) and the coordinates of Q are (–3, 4).

So, the coordinates of a point in a plane are unique.

5.2.2 Origin

1. The intersecting point of X-axis and Y-axis is called origin. We take origin as a reference point to locate other points in a plane. Origin is denoted by "O".

Example 1. State the abscissa and ordinate of the following points and describe the position of each point (i) P(8,8) (ii) Q (6,–8).

Solution : (i) P (8,8)

abscissa = 8 (x - coordinate); Ordinate = 8 (y - coordinate)

The point P is at a distance of 8 units from Y-axis measured along positive point of X-axis from origin. As its ordinate is 8, the point is at a distance of 8 units from X-axis measured along positive point of Y-axis from origin.

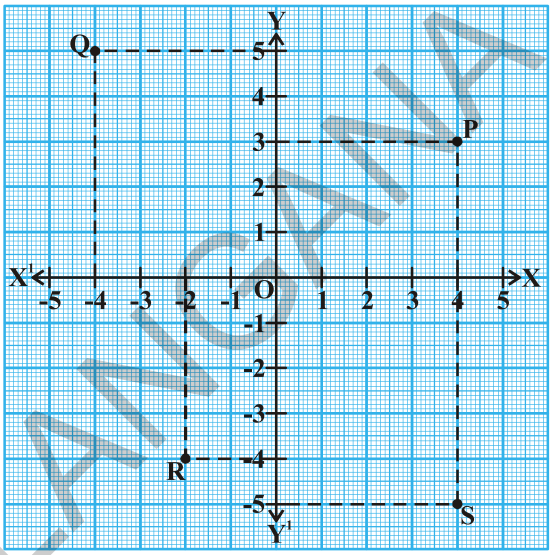
(ii) Q (6, –8)

abscissa = 6 ; Ordinate = –8

The point Q is at a distance of 6 units from Y-axis measured along positive X-axis and it is at a distance of 8 units from X-axis measured along negative Y-axis.

Example 2. Write the coordinates of the points marked in the graph.

Solution : 1. Draw a perpendicular line to X-axis from the point P. The perpendicular line touches X-axis at 4 units. Thus abscissa of P is 4. Similarly draw a perpendicular line to Y-axis from P. The perpendicular line touches Y-axis at 3 units. Thus ordinate of P is 3. Hence the Cordinates of P are (4, 3).



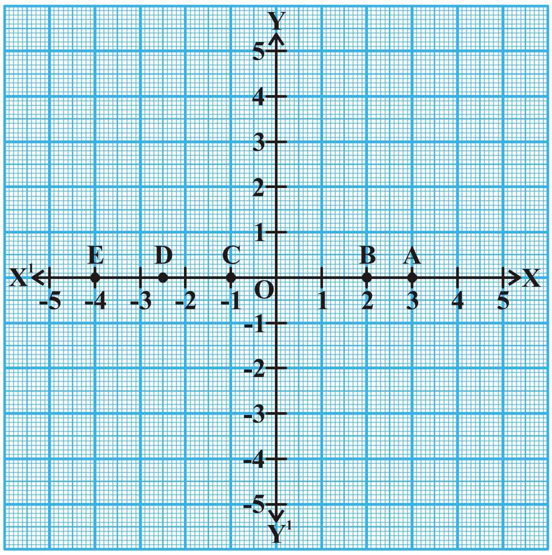
2. Similarly, the abscissa and ordinate of the point Q are –4 and 5 respectively. Hence the coordinates of Q are (–4, 5).

3. The abscissa and ordinate of the point R are –2 and –4 respectively. Hence the coordinates of R are (–2, –4).

4. The abscissa and ordinate of the point S are 4 and –5 respectively. Hence the coordinates of S are (4, –5).

Example-3. Write the coordinates of the points marked in the graph.

Solution : The point A is at a distance of 3 units from the Y-axis and at a distance zero units from the X-axis. Therefore the x coordinate of A is 3 and y-coordinate is 0. Hence the coordinates of A are (3,0).So think and discuss.



(i) The coordinates of B are (2,0). Why?

(ii) The coordinates of C are (–1,0). Why?

(iii) The coordinates of D are (–2.5, 0). Why?

(iv) The coordinates of E are (–4,0) why? What do you observe?

So as observed in figure, every point on the X-axis has no distance from X-axis. Therefore the y coordinate of a point lying on X-axis is always zero.

X-axis is denoted by the equation y = 0.

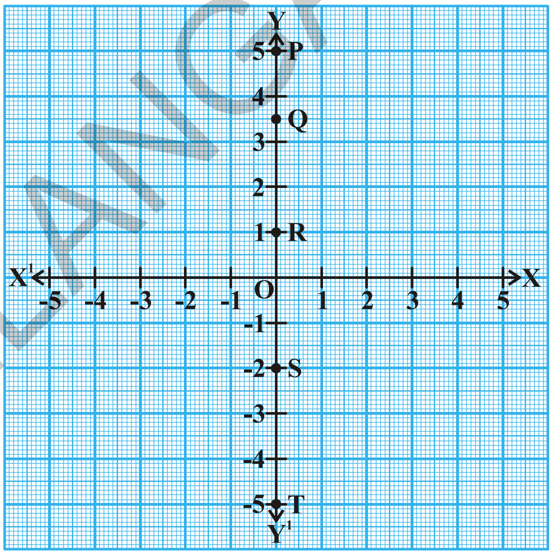
Do this

Among the points given below some of the points lie on X-axis. Identify them.

(i)	(0,5)	(ii)	(0,0)	(iii)	(3,0)
(iv)	(-5,0)	(v)	(-2,-3)	(vi)	(-6,0)
(vii)	(0,6)	(viii)	(0,a)	(ix)	(b,0)

Example-4. Write the coordinates of the points marked in graph.

Solution :



(i) The point P is at a distance of +5 units from the X-axis and at a distance zero from the Y-axis. Therefore the x-coordinate of P is 0 and y-coordinate is 5. Hence the coordinates of P are (0,5).

Think and discuss :

(ii) The coordinates of Q are (0, 3.5), why?

(iii) The coordinates of R are (0,1), why?

(iv) The coordinates of S are (0, –2), why?

(v) The coordinates of T are (0, –5), why?

Since every point on the Y-axis is at a zero distance from the Y-axis, the x-coordinate of the point lying on Y-axis is always zero. Y-axis is given by the equation $x = 0$.

5.2.3 Coordinates of Origin

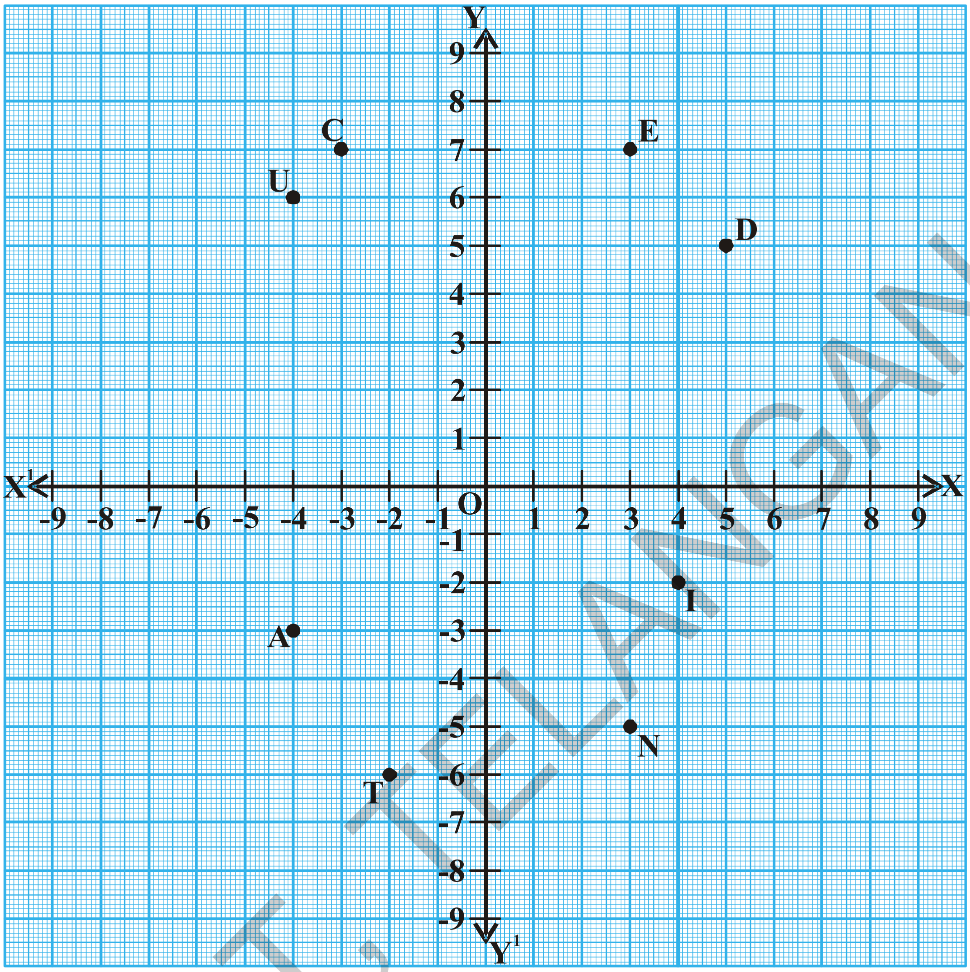
The point O lies on Y-axis. Its distance from Y-axis is zero. Hence its x-coordinate is zero. Also it lies on X-axis. Its distance from X-axis is zero. Hence its y-coordinate is zero.

Therefore the coordinates of the origin 'O' are (0, 0).

Try These

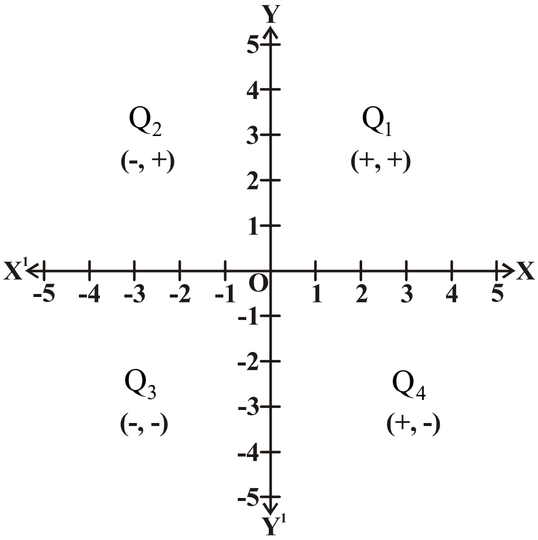
1. Which axis the points such as $(0, x)$ $(0, y)$ (0, 2) and (0, –5) lie on? Why ?
2. Write the general form of the points which lie on X-axis.

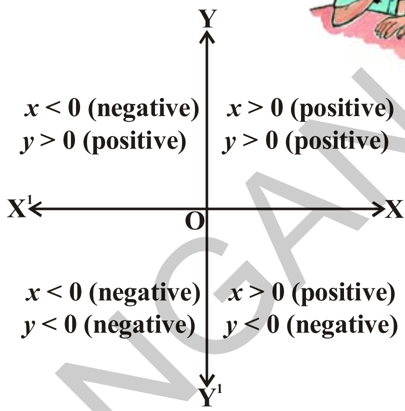
Example 5. Complete the table based on the following graph.



Point	Abscissa	Ordinate	Co-ordinates	Quadrant	Signs of co-ordinates
E	3	7	E (3,7)	Q_1	(+, +)
D	.....	.....	.....	.....	.....
U	−4	6	U (−4,6)	.....	(−,+)
C	.....	.....	.....	.....	.....
A	−4	−3	A (−4, −3)	.....	(−,−)
T	.....	.....	.....	.....	.....
I	4	−2	I (4, −2)	.....	(+,−)
O	.....	.....	.....	.....	.....
N	.....	.....	.....	.....	.....

From the above table you may have observed the following relationship between the signs of the coordinates of a point and the quadrant of a point in which it lies.





EXERCISE 5.2

1. Write the quadrant in which the following points lie.

   i) $(-2, 3)$ ii) $(5, -3)$ iii) $(4, 2)$ iv) $(-7, -6)$

   v) $(0, 8)$ vi) $(3, 0)$ vii) $(-4, 0)$ viii) $(0, -6)$

2. Write the abscissae and ordinates of the following points.

   i) $(4, -8)$ ii) $(-5, 3)$ iii) $(0, 0)$ iv) $(5, 0)$

   v) $(0, -8)$

   Note : Plural of abscissa is abscissae.

3. Which of the following points lie on the axes? Also name the axis.

   i) $(-5, -8)$ ii) $(0, 13)$ iii) $(4, -2)$ iv) $(-2, 0)$

   v) $(0, -8)$ vi) $(7, 0)$ vii) $(0, 0)$

4. Write the following based on the graph.

   i) The ordinate of L

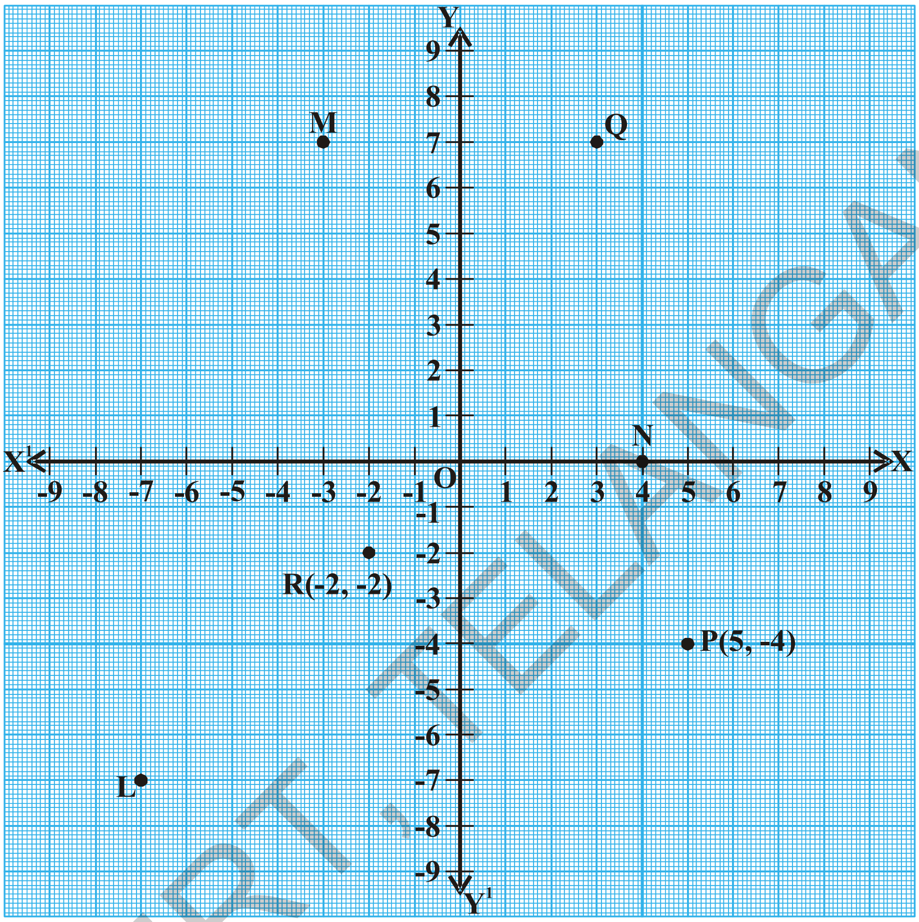
   ii) The ordinate of Q

   iii) $(-2, -2)$ is denoted by

iv) (5, –4) is denoted by

v) The abscissa of N

vi) The abscissa of M



5. State True or False, if 'false' write correct statement.

   i. In the Cartesian plane the horizontal line is called Y - axis.

   ii. In the Cartesian plane, the vertical line is called Y - axis.

   iii. The point which lies on both the axes is called origin.

   iv. The point (2, –3) lies in the third quadrant.

   v. (–5, –8) lies in the fourth quadrant.

   vi. The point (–x , –y) lies in the first quadrant where x < 0 , y < 0.

6. Plot the following ordered pairs on a graph sheet. What do you observe?

   i.. (1, 0), (3 , 0), (–2 , 0), (–5, 0), (0, 0), (5, 0), (–6, 0)

   ii. (0, 1), (0 , 3), (0 , –2), (0, –5), (0, 0), (0, 5), (0, –6)

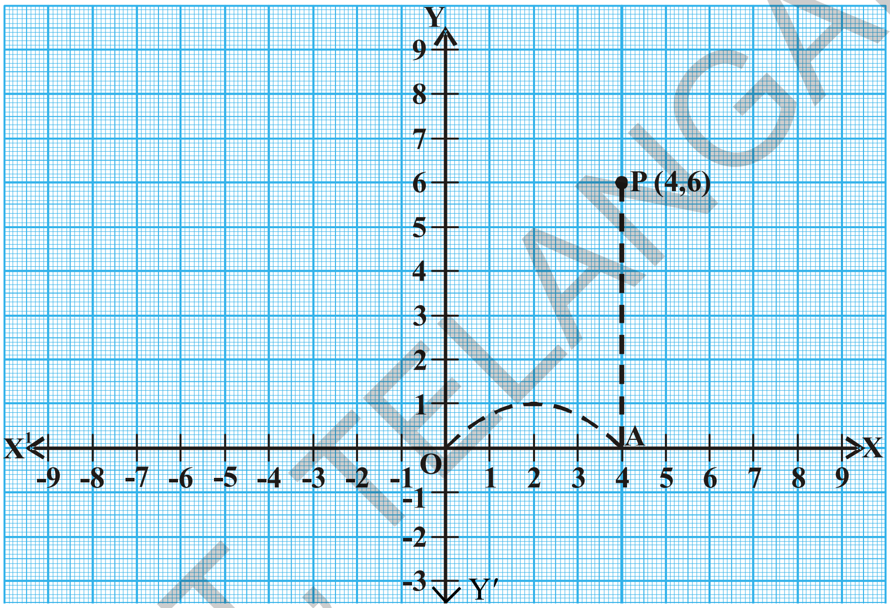
5.3 Plotting a point on the Cartesian plane (when its co-ordinates are given)

So far we have seen how to read the positions of points marked on a Cartesian plane. Now we shall learn to mark the point if its co-ordinates are given.

For instance how do you plot a point (4, 6)?

Can you say in which quardrant the point P lies?

We know that the abscissa (x-coordinate) is 4 and y-coordinate is 6.



$\therefore$ P lies in the first quadrant

The following process shall be followed in plotting the point P (4, 6) in the graph sheet.

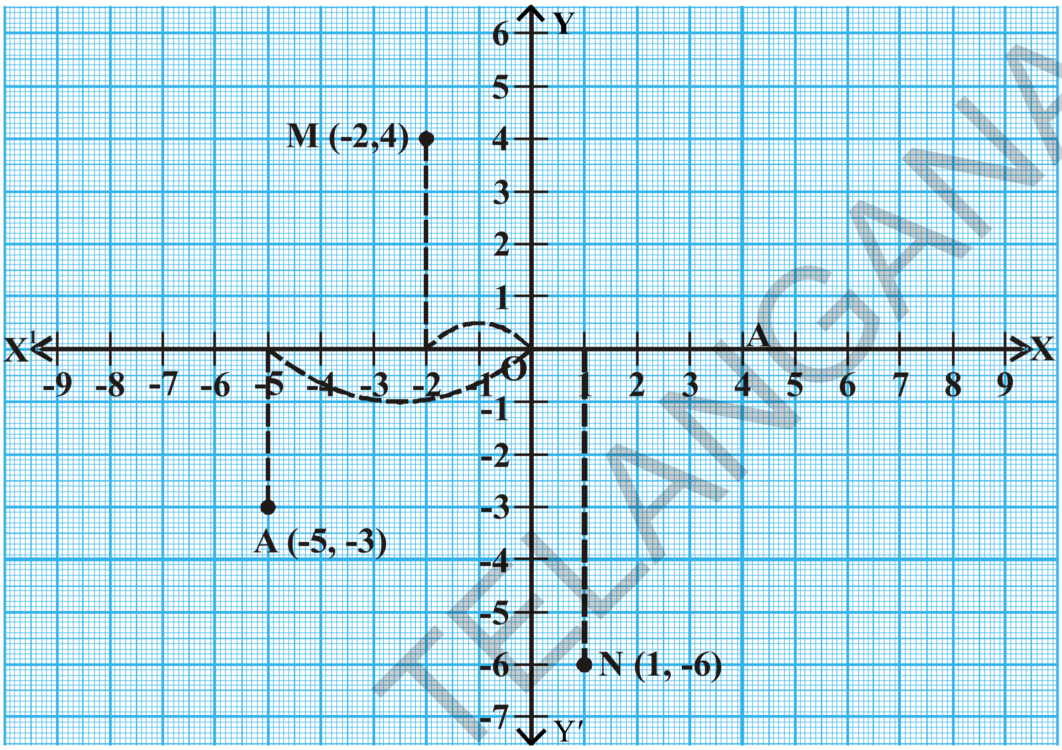
- Draw two number lines perpendicular to each other intersecting at their zeroes on a graph paper. Name the horizontal line as X-axis and the vertical line as Y-axis and name the intersecting point of both the lines as Origin 'O'.
- Keep the x-coordinate in mind, start from zero, i.e. from the Origin.
- Move 4 units along positive part of X-axis i.e. to its right side and mark the point A.
- From A move 6 units upward along the line parallel to positive part of Y-axis.
- Locate the position of the point 'P' as (4, 6).

The above process of marking a point on a Cartesian plane using their co-ordinates is called "plotting the point".

Example 6. Plot the following points in the Cartesian plane

(i) M (–2, 4), (ii) A (–5, -3), (iii) N (1, –6)

Solution : Draw the X-axis and Y-axis on the graph.



(i) Can you say in which quadrant the point M lie?

Since $x < 0$, $y >$ o, It lies in the second quadrant. Let us now locate its position.

M (–2,4) : start from the origin, move 2 units from zero along the negative part of X-axis i.e. on its left side.

From there move 4 units along the line parallel to positive Y-axis i.e. upwards.

Name it as M (–2,4).

(ii) A (–5, –3) :

The point A lies in the third quadrant. Start from zero, the Origin. Move 5 units from zero to its left side that is along the negative part of X-axis.

From there move 3 units along a line parallel to negative part of Y-axis i.e. downwards. Name it as A (–5, –3)

(iii) N (1, –6): In which quadrant does it lie?

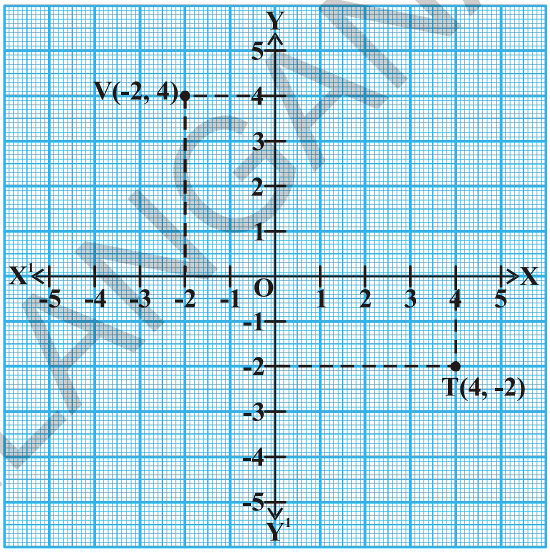
The point N lies in the fourth quadrant, start from zero i.e. origin. Move 1 unit along positive part of X-axis i.e. to the right side of zero. From there move 6 units along a line parallel to negative Y-axis i.e. downwards, and name the point as N (1, –6)

Do This

Plot the following points on a Cartesian plane.

1. B (−2, 3) 2. L (5, −8) 3. U (6, 4) 4. E (−3, −3)

Example 7 : Plot the points T(4, −2) and V(−2, 4) on a cartesian plane. Whether these two coordinates locate the same point?



Solution : In this example we plotted two points T (4, −2) and V(−2, 4)

Are the points (4, −2) and (−2, 4) distinct or same? Think.

We see that (4, −2) and (−2, 4) are at different positions. Repeat the above activity for the points P (8, 3), Q(3, 8) and A (4, −5), B(−5 , 4) and say whether the point (x, y) is different from (y, x) or not ?

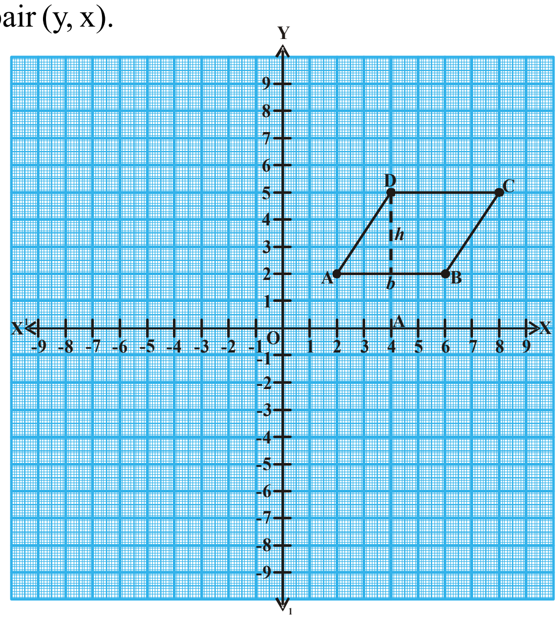
From the above plotting it is evident that the position of (x, y) in the Cartesian plane is different from the position of (y, x). i.e. the order of x and y is important in (x, y).

Therefore (x, y) is called an ordered pair.

If $x \neq y$, the ordered pair (x, y) $\neq$ ordered pair (y, x).

However if x = y, then (x, y) = (y, x)

Example 8: Plot the points A(2, 2), B(6, 2), C (8, 5) and D (4, 5) in a graph sheet. Join all the points to make it a parallelogram. Find the area of the parallelogram.



Solution: All the given points lie in Q_1.

from the graph b = AB = 4units.

height h = 3 units

Area of parallelogram

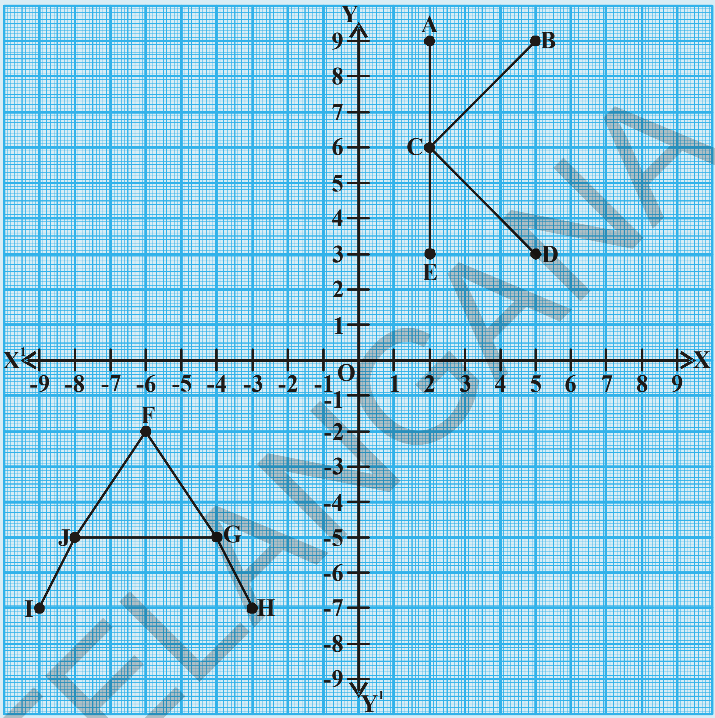
$= \text{base} \times \text{height}$

$= 4 \times 3 = 12 \text{ unit}^2$

Do This

(i) Write the coordinates of the points A, B, C, D, E.

(ii) Write the coordinates of F, G, H, I, J.



Exercise 5.3

1. Write the following points as a orderpair and plot in the Cartesian plane.

x	2	3	–1	0	–9	–4
y	–3	–3	4	11	0	–6
(*x*, *y*)						

2. Are the positions of (5, –8) and (–8, 5) same? Justify your answer.

3. What can you say about the position of the points (1, 2), (1, 3), (1, –4), (1, 0), and (1, 8). Locate on a graph sheet .

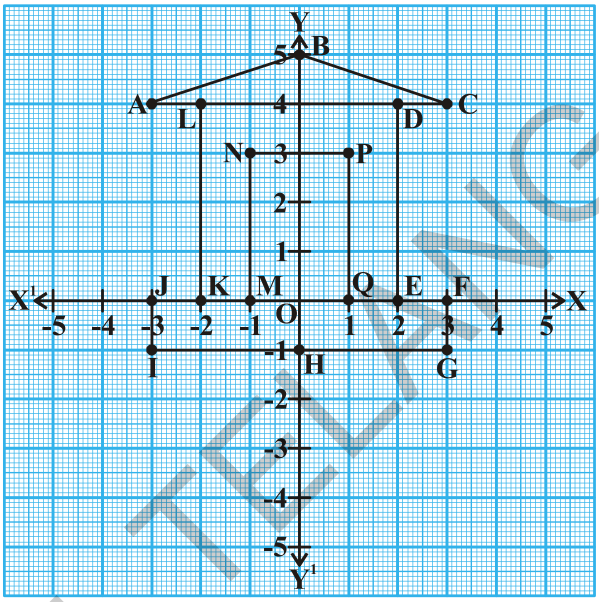
4. What can you say about the position of the points (5, 4), (8, 4), (3, 4), (0, 4), (–4, 4), (–2, 4)? Locate the points on a graph sheet. Justify your answer.

5. Plot the points (0, 0) (0, 3) (4, 3) (4, 0) in graph sheet. Join the points in order with straight lines to make a rectangle. Find the area of the rectangle thus formed.

6. Plot the points (2, 3), (6, 3) and (4, 7) in a graphsheet. Join them to make it a triangle. Find the area of the triangle thus formed.

7. Plot at least six points in a graph sheet, each having the sum of its coordinates equal to 5.
Hint : (–2, 7) (1, 4) ....

8. Look at the graph. Write the coordinates of the points A, B, C, D, E, F, G, H, I, J, K, L, L, M. N, P, O and Q.



9. In a graph Sheet Plot each pair of points, join them by line segments

i. (2, 5), (4, 7)	ii. (–3, 5), (–1, 7)	iii. (–3, –4), (2, –4)
iv. (–3, –5), (2, –5)	v. (4, –2), (4, –3)	vi. (–2, 4), (–2, 3)
vii. (–2, 1), (–2, 0)	viii. (4, 7), (4, –3)	ix. (4,–2), (2, –4)
x. (4, –3), (2,–5)	xi. (2, 5), (2, –5)	xii. (–3, 5), (–3, –5)
xiii. (–3, 5), (2, 5)	xiv. (–1, 7) (4, 7)	What do you observe

10. Plot the following pairs of points on the axes and join them with line segments.

(1, 0), (0, 9); (2, 0), (0, 8); (3, 0) (0, 7); (4, 0) (0, 6);

(5, 0) (0, 5); (6, 0) (0, 4); (7, 0) (0, 3); (8, 0) (0, 2); (9, 0) (0, 1).

What do you observe?

Activity

Study the positions of different cities like Hyderabad, New Delhi, Chennai and Vishakapatnam with respect to longitudes and latitudes on a globe.

Creative Activity

Take a graph sheet and plot the following pairs of points on it and join them in order with line segments.

(–9, 0), (–6, 4), (–2, 5), (2, 4), (5, 0) (–2, 0),

(–2, –8), (–3, –9), (–4, –8).

What do you notice ?

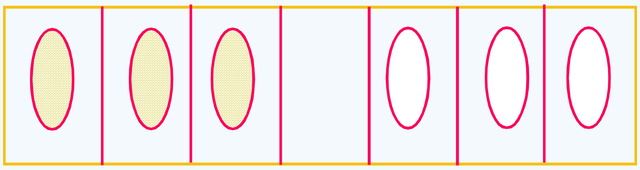
What have we discussed?

- We need two references to locate the exact position of a point in a plane.
- A point or an object in a plane can be located with the help of two perpendicular number lines. One of them is horizontal line (X-axis) and the other is vertical line (Y-axis).
- Cartesian plane is named after Rene Descartes.
- The point of intersection of X-axis and Y-axis is the orgin. Coordinates of orgin are (0, 0)
- The ordered pair (x, y) is different from the ordered pair (y, x).
- The equation of X-axis is given by y = 0.
- The equation of Y-axis is given by x = 0.

S9I4F2

Brain teaser

Look at the cards placed below you will find a puzzle



The white card pieces must change places with the shaded pieces while following these rules : (1) pieces of the same colour cannot jump over another (2) move one piece one space or jump at a time. Find the least number of moves.

Minimum number of moves is between 15 to 20. Can you do better?
To make the game more challanging, increase the number of pieces of cards

Chapter

6

Linear Equations in Two Variables

M1N4Y1

6.1 Introduction

We have come across many problems like

(i) If five pens cost ₹ 60, then find the cost of one pen.

(ii) A number when added to 7 gives 51. Find that number.

Here, in situation (i) the cost of the pen is unknown, while in situation (ii) the number is unknown. How do we solve questions of this type? We take letters *x*, *y* or *z* for the unknown quantities and write an equation for these situations.

For situation (i) we can write

$5 \times$ cost of a pen = ₹ 60

If the cost of a pen is ₹ y

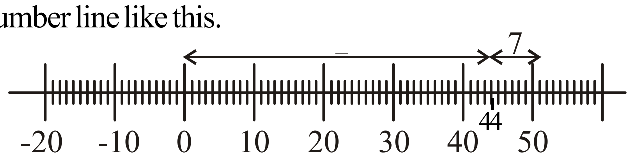
Then, $5 \times y = 60$ or $5y = 60$

Now solve it for y, we can find cost of a pen.

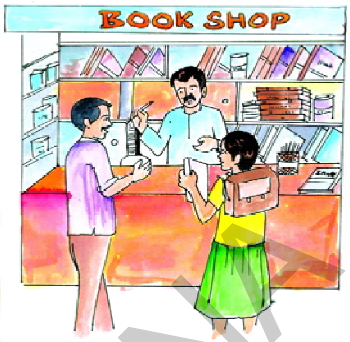
Likewise we can make an equation for situation (ii) and find the unknown number. Such type of equations are linear equations.

Equations like $x + 3 = 0$, $x + \sqrt{3} = 0$ and $\sqrt{2}x + 5 = 0$ are examples of linear equations in one variable. You also know that such equations have unique (implying one and only one) solution. You may also remember how to represent the solution on number line.

Hanif represented the solution of situation (ii) on the number line like this.



6.2 Linear Equations in Two Variables



Now consider the following situation :

One day Kavya went to a bookshop with her father to buy 4 notebooks and 2 pens. Her father paid ₹ 100 for all these.

Kavya did not know the cost of the note book and the pen separately. Now can you express this information in the form of an equation ?

Here, you can see that the cost of the single note book and also of the pen is unknown, i.e. there are two unknown quantities. Let us use x and y to denote them. So, the cost of a note book is ₹ x and the cost of a pen is ₹ y.

We represent the above information as an equation in the form $4x + 2y = 100$,

Have you observed the exponents of x and y in the above equation ?

Thus the above equation is in linear form with variables 'x' and 'y'.

If a linear equation has two variables then it is called a linear equation in two variables.

Therefore $4x + 2y = 100$ is an example of linear equation in two variables.

It is usually denoted by variables by 'x' and 'y'. But other letters may also be used.

$p + 3q = 50$, $\sqrt{3}u + \sqrt{2}v = \sqrt{11}$, $\frac{s}{2} - \frac{t}{3} = 5$ and $3 = \sqrt{5}x - 7y$ are examples of linear equation in two variables.

Note that you can put the above equations in the form of $p + 3q - 50 = 0$, $\sqrt{3}u + \sqrt{2}v - \sqrt{11} = 0$, $\frac{s}{2} - \frac{t}{3} - 5 = 0$ and $\sqrt{5}x - 7y - 3 = 0$ respectively.

Therefore the general form of linear equation in two variables x, y is $ax + by + c = 0$. Where a, b, c are real numbers, and a, b are not simultaneously zero.

Example 1. Sachin and Sehwag scored 137 runs together. Express the information in the form of linear equation in two variables.

Solution : Let runs scored by Sachin be 'x' and runs scored by Sehwag be 'y' .

Then the above information in the form of an equation is

$$x + y = 137$$

Example 2. Hema's age is 4 times the age of Mary. Write a linear equation in two variables to represent this information.

Solution : Let Hema's age be 'x' years and of Mary be 'y' years,

Given Hema's age = 4 times Mary's age

That is $x = 4y$

$\Rightarrow x - 4y = 0$ (how?)

Example 3. A number is 27 more than the number obtained by reversing its digits. If its unit's and ten's digits are x and y respectively, write the linear equation representing the above statement.

Solution : Units digit is represented by x and tens digit by y, then the number is 10y + x

If we reverse the digits then the new number would be 10x + y (Recall the place value of digits in a two digit number).

Therefore according to the given condition

(two digit number) – (number formed by reversing the digits) = 27.

i.e., $10y + x - (10x + y) = 27$

$\Rightarrow 10y + x - 10x - y - 27 = 0$

$\Rightarrow 9y - 9x - 27 = 0$

$\Rightarrow y - x - 3 = 0$

$\Rightarrow x - y + 3 = 0$ is the required equation.

Example 4. Express each of the following equations in the form of ax + by + c = 0 and write the values of a, b and c.

i) $3x + 4y = 5$ ii) $x - 5 = \sqrt{3}y$

iii) $3x = y$ iv) $\frac{x}{2} + \frac{y}{2} = \frac{1}{6}$

v) $3x - 7 = 0$

Solution : (i) $3x + 4y = 5$ can be written as

$3x + 4y - 5 = 0.$

Here $a = 3$, $b = 4$ and $c = -5$.

(ii) $x - 5 = \sqrt{3}y$ can be written as

$1.x - \sqrt{3}y - 5 = 0.$

Here $a = 1, b = -\sqrt{3}$ and $c = -5.$

(iii) The equation $3x = y$ can be written as

$3x - y + 0 = 0.$

Here $a = 3, b = -1$ and $c = 0.$

(iv) The equation $\frac{x}{2} + \frac{y}{2} = \frac{1}{6}$ can be written as

$\frac{x}{2} + \frac{y}{2} - \frac{1}{6} = 0;$

Here $a = \frac{1}{2}, b = \frac{1}{2}$ and $c = \frac{-1}{6}$

(v) $3x - 7 = 0$ can be written as

$3x + 0.y - 7 = 0.$

Here $a = 3, b = 0; c = -7$

Example-5. Write each of the following in the form of $ax + by + c = 0$ and find the values of a, b and c

i) $x = -5$

ii) $y = 2$

iii) $2x = 3$

iv) $5y = -3$

Solution :

S.No.	Given equation	Expressed as $ax + by + c = 0$	Value of a, b, c		
			a	b	c
1	$x = -5$	$1 \cdot x + 0 \cdot y + 5 = 0$	1	0	5
2	$y = 2$	$0 \cdot x + 1 \cdot y - 2 = 0$	0	1	–2
3	$2x = 3$	---	---	---	---
4	$5y = -3$	----	----	---	---

Try This

1. Express the following linear equations in the form of ax + by + c = 0 and write the values of a, b, c in each case.

   i) $3x + 2y = 9$ ii) $-2x + 3y = 6$ iii) $9x - 5y = 10$

   iv) $\frac{x}{2} - \frac{y}{3} - 5 = 0$ v) $2x = y$

Exercise - 6.1

1. Express the following linear equations in the form of ax+by+c=0 and indicate the values of a, b and c in each case.

   i) $8x + 5y - 3 = 0$ ii) $28x - 35y = -7$ iii) $93x = 12 - 15y$

   iv) $2x = -5y$ v) $\frac{x}{3} + \frac{y}{4} = 7$ vi) $y = \frac{-3}{2}x$

   vii) $3x + 5y = 12$

2. Write each of the following in the form of ax + by + c = 0 and find the values of a, b and c

   i) $2x = 5$ ii) $y - 2 = 0$ iii) $\frac{y}{7} = 3$ iv) $x = \frac{-14}{13}$

3. Express the following statements as a linear equation in two variables.

   (i) The sum of two numbers is 34.

   (ii) The cost of a ball pen is ₹ 5 less than half the cost of a fountain pen.

   (iii) Bhargavi got 10 more marks than double the marks of Sindhu.

   (iv) The cost of a pencil is ₹ 2 and a ball point pen is ₹ 15. Sheela pays ₹ 100 for the pencils and pens she purchased.

   (v) Yamini and Fatima of class IX together contributed ₹200/- towards the Prime Minister's Relief Fund.

   (vi) The sum of a two digit number and the number obtained by reversing the order of its digits is 121. If the digits in unit's and ten's place are 'x' and 'y' respectively.

6.3 Solution of a Linear Equation in two variables

You know that linear equation in one variable has a unique solution.

What is the solution of the equation $3x - 4 = 8$?

Consider the equation $3x - 2y = 5$.

What can we say about the solution of this linear equation in two variables? Do we have only one value in the solution or do we have more ? Let us explain.

Can you say $x = 3$ is a solution of this equation?

Let us check, if we substitute $x = 3$ in the equation

We get $3\ (3) - 2y = 5$

$$9 - 2y = 5$$

i.e., Still we cannot find the solution of the given equation. So, to know the solution, besides the value of 'x' we also need the value of 'y'. we can get value of y from the above equation $9 - 2y = 5$. $\Rightarrow$ $2y = 4$ or $y = 2$

The values of x and y which satisfy the equation $3x - 2y = 5$, are $x = 3$ and $y = 2$. Thus to statisfy, a linear equation in two variables we need two values, one value for 'x' and one value for y.

Therefore any pair of values of 'x' and 'y' which satisfy the linear equation in two variables is called its solution.

We observed that $x = 3, y = 2$ is a solution of $3x - 2y = 5$. This solution is written as an ordered pair (3, 2), first writing the value for 'x' and then the value for 'y'. Are there any other solutions for the equation? Pick a value of your choice say $x = 4$ and substitute it in the equation $3x - 2y = 5$. Then the equation reduces to $12 - 2y = 5$. Which is an equation in one variable. On solving this we get.

$$y = \frac{12-5}{2} = \frac{7}{2}, \quad \text{so} \quad \left(4, \frac{7}{2}\right) \text{ is another solution, of } 3x - 2y = 5$$

Do you find some more solutions for $3x - 2y = 5$? Check if $(1, -1)$ is another solution?

Thus for a linear equation in two variables we can find many solutions.

Note : An easy method of getting two solutions is put $x = 0$ and get the corresponding value of 'y'. Similarly we can put $y = 0$ and obtain the corresponding value of 'x'.

Try This

Find 5 more pairs of values that are solutions for the above equation.

Example 6. Find four different solutions of $4x + y = 9$. (Complete the table wherever necessary)

Solution :

S.No.	Choice of a value for variable x or y	Simplification		Solution
1.	$x = 0$	$4x + y = 9$	$\Rightarrow 4 \times 0 + y = 9$ $\Rightarrow y = 9$	(0,9)
2.	$y = 0$	$4x + y = 9$	$\Rightarrow 4x + 0 = 9$ $\Rightarrow 4x = 9$ $\Rightarrow x = 9/4$	$\left(\frac{9}{4}, 0\right)$
3.	$x = 1$	$4x + y = 9$	$\Rightarrow 4 \times 1 + y = 9$ $\Rightarrow 4 + y = 9$ $\Rightarrow y = 5$	——
4.	$x = -1$		———	$(-1, 13)$

$\therefore$ $(0, 9), \left(\frac{9}{4}, 0\right), (1, 5)$ and $(-1, 13)$ are some of the solutions for the above equation.

Example-7. Check which of the following are solutions of an equation $x + 2y = 4$? (Complete the table wherever necessary)

i) $(0, 2)$ ii) $(2, 0)$ iii) $(4, 0)$ (iv) $(\sqrt{2}, -3\sqrt{2})$

v) $(1, 1)$ vi) $(-2, 3)$

Solution : We know that if we get LHS = RHS when we substitute a pair in the given equation, then it is a solution.

The given equation is $x + 2y = 4$

S. No	Pair of Values	Value of LHS	Value of RHS	Relation between LHS and RHS	Solution/ not Solution
1.	(0, 2)	$x + 2y = 0 + (2 \times 2)$ $= 0 + 4 = 4$	4	$\therefore$ LHS=RHS	$\therefore$ (0, 2) is a solution
2.	(2, 0)	$x + 2y = 2 + (2 \times 0)$ $= 2 + 0 = 2$	4	......	(2, 0) is not a a solution
3.	(4, 0)	$x + 2y = 4 + (2 \times 0)$ $= 4 + 0 = 4$	4	LHS = RHS	___
4.	$(\sqrt{2}, -3\sqrt{2})$	$x + 2y = \sqrt{2} + 2(-3\sqrt{2})$ $= \sqrt{2} - 6\sqrt{2}$ $= -5\sqrt{2}$	___	LHS $\neq$ RHS	$(\sqrt{2}, -3\sqrt{2})$ Not a solution
5.	(1, 1)	___	4	LHS $\neq$ RHS	(1, 1) Not a solution
6.	___	$x + 2y = -2 + (2 \times 3)$ $= -2 + 6 = 4$	4	LHS = RHS	(−2, 3) is a solution

Example-8. If $x = 3, y = 2$ is a solution of the equation $5x - 7y = k$, find the value of k and write the resultant equation.

Solution : If $x = 3, y = 2$ is a solution of the equation

$$5x - 7y = k \text{ then } 5 \times 3 - 7 \times 2 = k$$
$$\Rightarrow 15 - 14 = k$$
$$\Rightarrow 1 = k$$
$$\therefore k = 1$$

The resultant equation is $5x - 7y = 1$.

Example-9. If $x = 2k + 1$ and $y = k$ is a solutions of the equation $5x + 3y - 7 = 0$, find the value of k.

Solution : It is given that $x = 2k + 1$ and $y = k$ is a solution of the equation $5x + 3y - 7 = 0$ by substituting the value of x and y in the equation we get.

$\Rightarrow 5(2k + 1) + 3k - 7 = 0$

$\Rightarrow 10k + 5 + 3k - 7 = 0$

$\Rightarrow 13k - 2 = 0$ (this is the linear equation in one variable).

$\Rightarrow 13k = 2$

$\therefore k = \dfrac{2}{13}$

EXERCISE - 6.2

1. Find three different solutions of the each of the following equations.

   i) $3x + 4y = 7$ ii) $y = 6x$ iii) $2x - y = 7$

   iv) $13x - 12y = 25$ v) $10x + 11y = 21$ vi) $x + y = 0$

2. If (0, a) and (b, 0) are the solutions of the following linear equations. Find 'a' and 'b'.

   i) $8x - y = 34$ ii) $3x = 7y - 21$ iii) $5x - 2y + 3 = 0$

3. Check which of the following are solutions of the equation $2x - 5y = 10$

   i) (0, 2) ii) (0, –2) iii) (5, 0) iv) $(2\sqrt{3}, -\sqrt{3})$ v) $\left(\frac{1}{2}, 2\right)$

4. Find the value of k, if $x = 2, y = 1$ is a solution of the equation $2x + 3y = k$. Find two more solutions of the resultant equation.

5. If $x = 2 - \alpha$ and $y = 2 + \alpha$ is a solution of the equation $3x - 2y + 6 = 0$ find the value of 'α'. Find three more solutions of the equation.

6. If $x = 1, y = 1$ is a solution of the equation $3x + ay = 6$, find the value of 'a'.

7. Write five different linear equations in two variables and find three solutions for each of them?

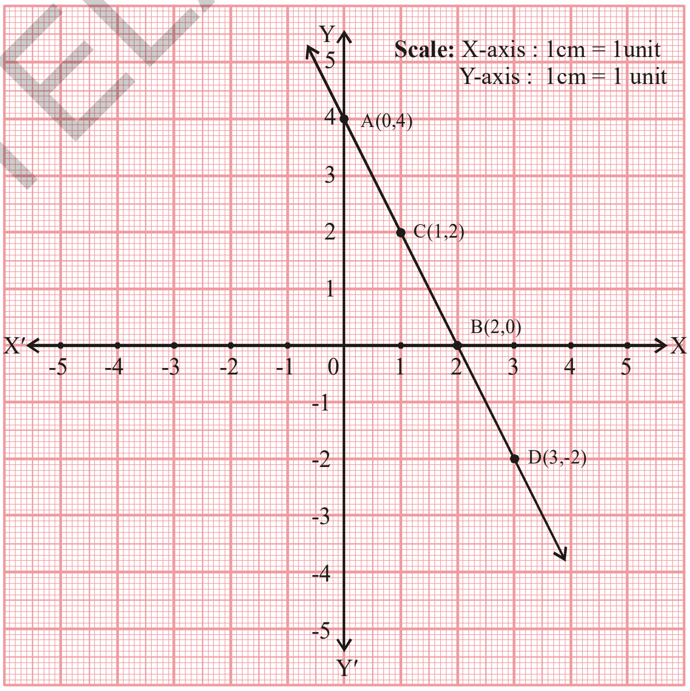
6.4 Graph of a Linear Equation in Two Variables

We have learnt that each linear equation in two variables has many solutions. If we take possible solutions of a linear equation, can we represent them on the graph? We know each solution is a pair of real numbers that can be expressed as a point in the graph.

Consider the linear equation in two variables $4 = 2x + y$. It can also be expressed as $y = 4 - 2x$. For this equation we can find the value of 'y' for a particular value of x. For example if $x = 2$ then $y = 0$. Therefore (2, 0) is a solution. In this way we find as many solutions as we can. Write all these solutions in the following table by writing the value of 'y' against the corresponding value of x.

Table of solutions:

x	$y = 4 - 2x$	(x, y)
0	$y = 4 - 2(0) = 4$	(0, 4)
2	$y = 4 - 2(2) = 0$	(2, 0)
1	$y = 4 - 2(1) = 2$	(1, 2)
3	$y = 4 - 2(3) = -2$	(3, –2)



We see for each value of x there is one value of y. Let us take the value of 'x' along the X-axis. and take the value of y along the Y-axis. Let us plot the points (0, 4), (2, 0), (1, 2) and (3, -2) on the graph. If we join any of these two points we obtain a straight line AD.

Do all the other points also lie on the line AD?

Now pick any other point on the line say (4,–4). Is this a solution?

If $x = 0$;
$y = 4 - 2x = 4 - 2(0) = 4$
If $x = 2$
$y = 4 - 2(2) = 0$

Pick up any other point on this line $\overleftrightarrow{AD}$ and check if its coordinates satisfy the equation or not?

Now take any point not on the line $\overleftrightarrow{AD}$ say (1, 1). Does it satisfy the equation?

Can you find any point that is not on the line AD but satisfies the equation?

Let us list our observations:

1. Every solution of the linear equation represents a point on the line.
2. Every point on this line is a solution of the linear equation.
3. Any point that does not lie on this line is not a solution of the equation and vice a versa.
4. The collection of points that give the solution of the linear equation is the graph of the linear equation.

We notice that the graphical representation of a linear equation in two variables is a straight line. Thus, $ax + by + c = 0$(a and b are not simultaneously zero) is called a linear equation in two variables.

6.4.1 How to draw the graph of a linear equation

Steps :

1. Write the linear equation.
2. Put $x = 0$ in the given equation and find the corresponding value of y.
3. Put $y = 0$ in the given equation and find the corresponding value of 'x'.
4. Write the values of x, y of steps (2), (3) as point (x, y).
5. Plot the points on the graph paper.
6. Join these points.

Thus line drawn is the graph of linear equation in two variables. However to check the correctness of the line it is better to take more than two points. To find more solutions take different values for 'x' substitute them in the given equation and find the corresponding values of 'y'.

TRY THESE

Take a graph paper, plot the point (2, 4), and draw a line passing through it. Now answer the following questions.

1. Can you draw another line that passes through the point (2, 4)?
2. How many such lines can be drawn?
3. How many linear equations in two variables exist for which (2, 4) is a solution?

Example-10. Draw the graph of the equation $y - 2x = 4$ and then answer the following.

(i) Does the point (2, 8) lie on the line? Is (2, 8) a solution of the equation? Check by substituting (2, 8) in the equation.

(ii) Does the point (4, 2) lie on the line? Is (4, 2) a solution of the equation? Check algebraically also.

(iii) From the graph find three more solutions of the equation and also three more which are not solutions.

Solution : Given $y - 2x = 4 \Rightarrow y = 2x + 4$

Table of Solutions

x	$y = 2x + 4$	(x, y)	Point
0	$y = 2(0) + 4 = 4$	(0, 4)	A(0, 4)
–2	$y = 2(-2)+4 = 0$	(–2, 0)	B(–2, 0)
1	$y = 2(1) + 4 = 6$	(1, 6)	C(1, 6)

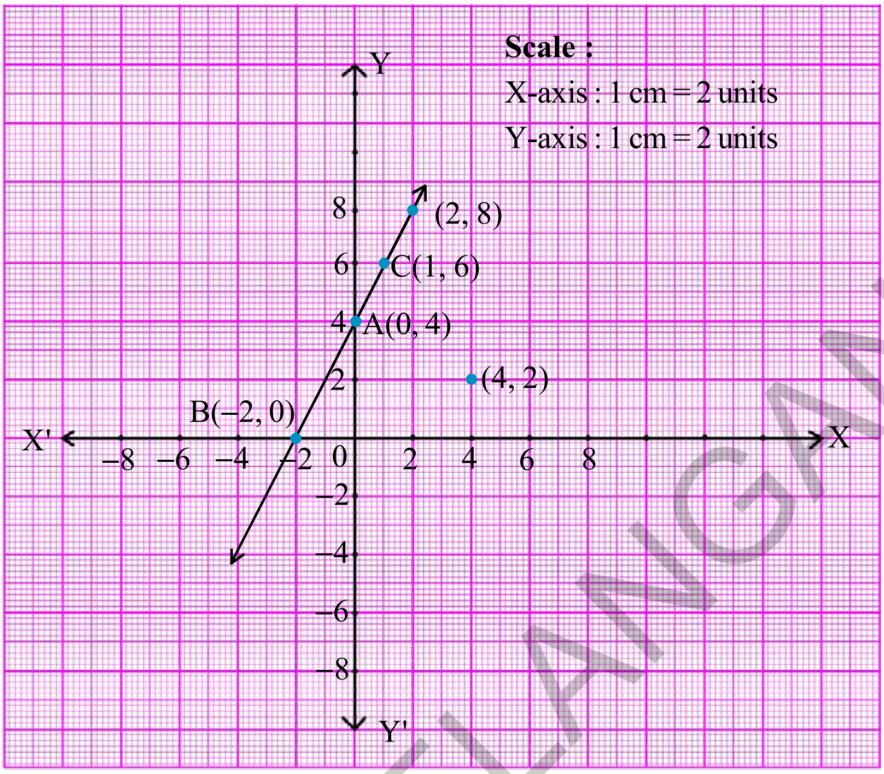
Plotting the points A, B and C on the graph paper and join them to get the straight line BC as shown in graph sheet. This line is the required graph of the equation $y - 2x = 4$.

(i) Plot the point (2, 8) on the graph paper. From the graph it is clear that the point (2, 8) lies on the line $\overleftrightarrow{BC}$.

Checking algebraically: On substituting (2, 8) in the given equation, we get

$$\text{LHS} = y - 2x = 8 - 2x2 = 8 - 4 = 4 = \text{RHS},$$

So (2, 8) is a solution



(ii) Plot the point (4, 2) on the graph paper. You find that (4, 2) does not lie on the line $\overleftrightarrow{BC}$.

Checking algebraically: By substituting (4, 2) in the given equation we have

$$\text{LHS} = y - 2x = 2 - 2\times4 = 2 - 8 = -6 \neq \text{RHS, so } (4, 2) \text{ is not a solution.}$$

(iii) We know that every point on the line is the solution of the equation. (–4,–4), (–3,–2) and (–1,2) are the solutions of the given equation y=2x+4, Where as (1,5), (2,1) and (–4,1) are not the solutions of the given equation, since these three points do not lie on the line in the graph.

Example-11. Draw the graph of the equation $x - 2y = 3$.

From the graph find (i) The solution (x, y) where $x = -5$

(ii) The solution (x, y) where $y = 0$

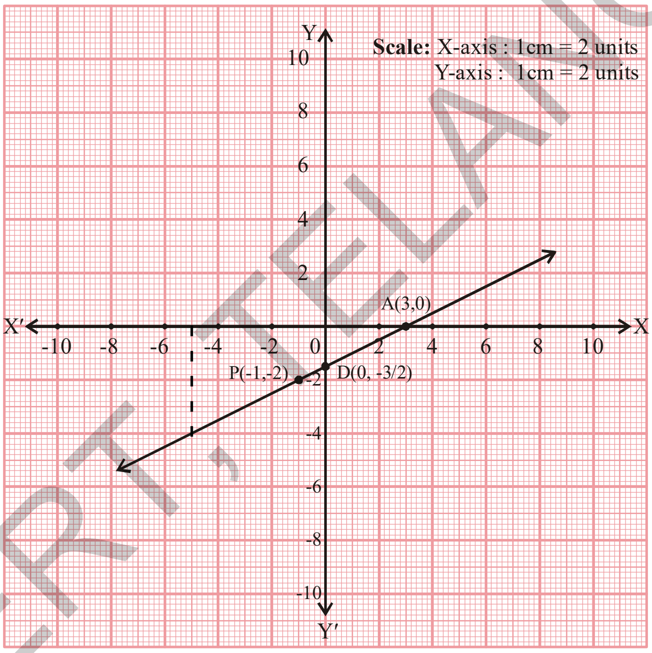
(iii) The solution (x, y) where $x = 0$

Solution : We have $x - 2y = 3 \Rightarrow y = \dfrac{x-3}{2}$

Table of Solutions

x	$y = \frac{x-3}{2}$	(x, y)	Point
3	$y = \frac{3-3}{2} = 0$	(3, 0)	A
1	$y = \frac{1-3}{2} = -1$	(1, – 1)	B
– 1	$y = \frac{-1-3}{2} = -2$	(– 1, – 2)	C

Plotting the points A, B, C on the graph paper and on joining them we get a straight line as shown in the following figure. This line is the required graph of the equation $x - 2y = 3$



(i) We have to find a solution (x, y) where $x = -5$, that is we have to find a point which lies on the straight line and whose x-coordinate is '-5'. To find such a point we draw a line parallel to y-axis at $x = -5$. (in the graph it is shown as dotted line). This line meets the graph at 'P' from there we draw another line parallel to X-axis meeting the Y-axis at $y = -4$.

The coordinates of $P = (-5, -4)$

Since $P(-5, -4)$ lies on the straight line $x - 2y = 3$, it is a solution.

(ii) We have to find a solution (x, y) where $y = 0$.

Since $y = 0$, this point (x, 0) lies on the X-axis. Therefore we have to find a point that lies on the X-axis and on the graph of $x - 2y = 3$.

From the graph it is clear that (3, 0) is the required point.

Therefore, the solution is (3, 0).

(iii) We have to find a solution (x, y) where x = 0.

Since x = 0 this point (0, y) lies on the Y-axis. Therefore we have to find a point that lies on the Y-axis and on the graph of $x - 2y = 3$.

From the graph it is clear that $\left(0, \frac{-3}{2}\right)$ is this point.

Therefore, the solution is $\left(0, \frac{-3}{2}\right)$.

Exercise - 6.3

1. Draw the graph of each of the following linear equations.

   i) $2y = -x + 1$ ii) $-x + y = 6$ iii) $3x + 5y = 15$ iv) $\frac{x}{2} - \frac{y}{3} = 3$

2. Draw the graph of each of the following linear equations and answer the questions.

   i) $y = x$ ii) $y = 2x$ iii) $y = -2x$ iv) $y = 3x$ v) $y = -3x$

   i) Are all these equations of the form $y = mx$, where m is a real number?

   ii) Are all these graphs passing through the origin?

   iii) What can you conclude about these graphs?

3. Draw the graph of the equation $2x + 3y = 11$. Find the value of y when x = 1 from the graph.

4. Draw the graph of the equation $y - x = 2$. Find from the graph

   i) the value of y when x = 4

   ii) the value of x when y =-3

5. Draw the graph of the equation 2x+3y=12. Find the solutions from the graph

   i) Whose y-coordinate is 3

   ii) Whose x-coordinate is -3

6. Draw the graph of each of the equations given below and also find the coordinates of the points where the graph cuts the coordinate axes

   i) $6x - 3y = 12$ ii) $-x + 4y = 8$ iii) $3x + 2y + 6 = 0$

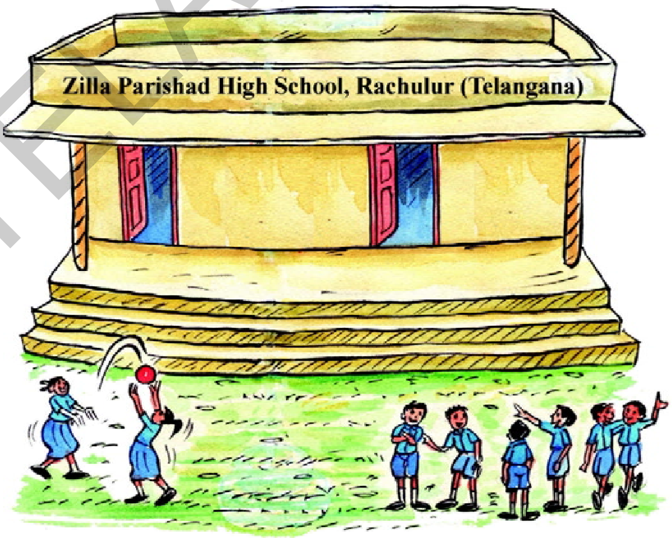
7. Rajiya and Preethi two students of Class IX together collected ₹ 1000 for the Prime Minister Relief Fund for victims of natural calamities. Write a linear equation and draw a graph to depict the statement.
8. Gopaiah sowed wheat and paddy in two fields of total area 5000 square meters. Write a linear equation and draw a graph to represent the same?
9. The force applied on a body of mass 6 kg. is directly proportional to the acceleration produced in the body. Write an equation to express this observation and draw the graph of the equation.
10. A stone is falling from a mountain. The velocity of the stone is given by V = 9.8t. (t = time) Draw its graph and find the velocity of the stone '4' seconds after start.

Example-12. 25% of the students in a school are girls and others are boys. Form a linear equation in two variables and draw a graph for this. By observing the graph, answer the following :

(i) Find the number of boys, if the number of girls is 25.

(ii) Find the number of girls, if the number of boys is 45.

(iii) Take three different values for number of boys and find the number of girls. Similarly take three different values for number of girls and find the number of boys?



Solution : Let the number of girls be 'x' and number of boys be 'y', then

Total number of students = x + y

According to the given information

Number of girls = 25% of the total number of students

$$x = 25\% \text{ of } (x + y)$$

$$= \frac{25}{100} \times (x + y) = \frac{1}{4}(x + y)$$

$$x = \frac{1}{4}(x + y)$$

$$4x = x + y$$

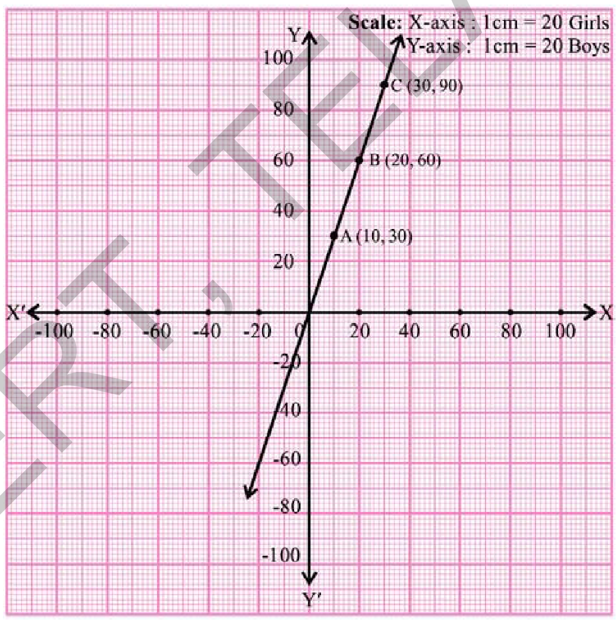
$$3x = y$$

The required equation is $3x = y$ or $3x - y = 0$.

Table of Solutions

x	y = 3x	(x, y)	Point
10	30	(10, 30)	A
20	60	(20, 60)	B
30	90	(30, 90)	C

Plotting the points A, B and C on the graph and on joining them we get the straight line as shown in the following figure.



From the graph we find that

(i) If the number of girls is 25 then the number of boys is 75.

(ii) If the number of boys is 45, then the number of girls is 15.

(iii) Choose the number you want for girls and find the corresponding number of boys. Similarly choose the numbers you want for boys and find the corresponding number of girls. Here observe the graph and equation. Graph of equation $y = 3x$ passes through origin if the equation is in the form $y = mx$ where m is a real number the line passes through the origin.

Example-13: For each graph given below, four linear equations are given. Out of these find the equation that represents the given graph.

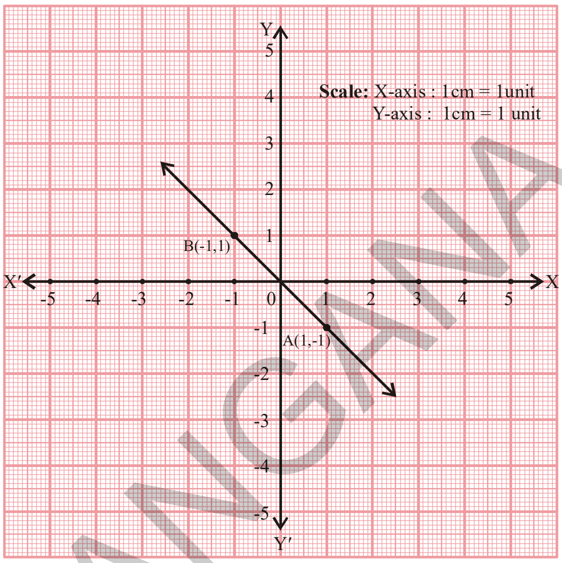
(i) Equations are

A) $y = x$

B) $x + y = 0$

C) $y = 2x$

D) $2 + 3y = 7x$



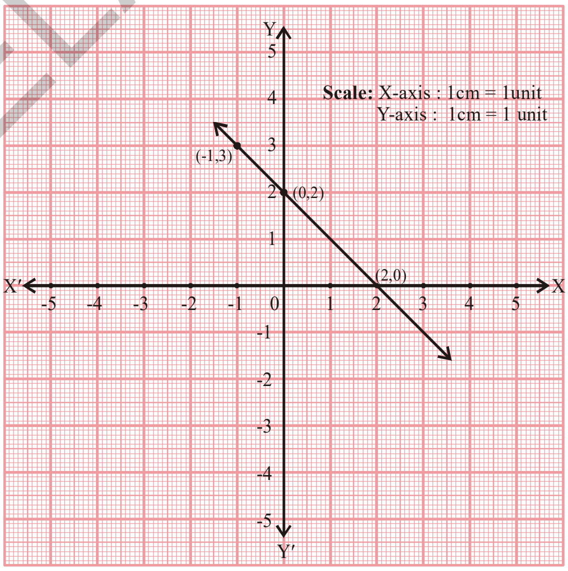
(ii) Equations are

A) $y = x + 2$

B) $y = x - 2$

C) $y = -x + 2$

D) $x + 2y = 6$



Solution :

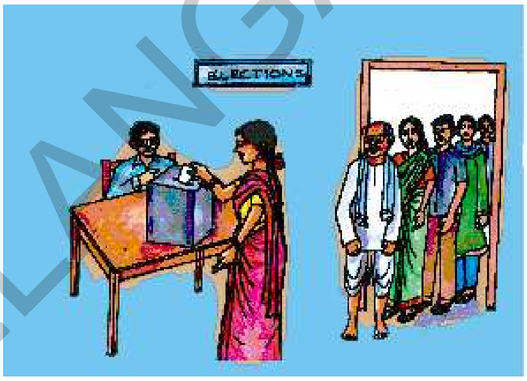
(i) From the graph we see $(1, -1)$ $(0, 0)$ $(-1, 1)$ lie on the same line. So these are the solutions of the required equation i.e. if we substitute these points in the required equation it should be satisfied. So, we have to find an equation that should be satisfied by these pairs. If we substitute $(1, -1)$ in the first equation $y = x$ it is not satisfied. So $y = x$ is not the required equation.
Putting $(1, -1)$ in $x + y = 0$ we find that it satisfies the equation. In fact all the three points satisfy the second equation. So $x + y = 0$ is the required equation.

We now check whether $y = 2x$ and $2 + 3y = 7x$ are also satisfied by $(1, -1)$ $(0, 0)$ and $(-1, 1)$. We find they are not satisfied. So, they are not the required equations.

(ii) The points on the line are $(2, 0)$, $(0, 2)$ and $(-1, 3)$. All these points don't satisfy the first and second equation. Let us take the third equation $y = -x + 2$. If we substitute the above three points in the equation, it is satisfied. So required equation is $y = -x + 2$. Check whether these points satisfies the equation $x + 2y = 6$?

EXERCISE - 6.4

1. In a election 60% of voters cast their votes. Form an equation and draw the graph for this data. Find the following from the graph.
   (i) The total number of voters, if 1200 voters cast their votes
   (ii) The number votes cast, if the total number of voters are 800



[**Hint:** If the number of voters who cast their votes be 'x' and the total number of voters be 'y' then $x = 60\%$ of y.]

2. When Rupa was born, her father was 25 years old. Form an equation and draw a graph for this data. From the graph find
   (i) The age of the father when Rupa is 25 years old.
   (ii) Rupa's age when her father is 40 years old.
3. An auto charges ₹ 15 for first kilometer and ₹ 8 each for each subsequent kilometer. For a distance of 'x' km. an amount of ₹ 'y' is paid.
   Write the linear equation representing this information and draw the graph. With the help of graph find the distance travelled if the fare paid is ₹ 55? How much should be paid for 7 kilometers?
4. A lending library has fixed charge for the first three days and an additional charges for each day thereafter. John paid ₹27 for a book kept for seven days. If the fixed charges be ₹x and subsequent per day charges be ₹y, then write the linear equation representing the above information and draw the graph of the same. From the graph, find fixed charges for the first three days if additional charges for each day thereafter is ₹4. Find additional charges for each day thereafter if the fixed charges for the first three days is ₹7.

5. The parking charges of a car in Hyderabad Railway station for first two hours is ₹ 50 and ₹10 for each subsequent hour. Write down an equation and draw the graph. Find the following charges from the graph

 (i) For three hours (ii) For six hours

 (iii) How many hours did Rekha park her car if she paid ₹ 80 as parking charges?

6. Sameera was driving a car with uniform speed of 60 kmph. Draw distance-time graph. From the graph find the distance travelled by Sameera in

 (i) $1\frac{1}{2}$ hours (ii) 2 hours (iii) $3\frac{1}{2}$ hours

7. The ratio of molecular weight of Hydrogen and Oxygen in water is 1:8. Set up an equation between Hydrogen and Oxygen and draw its graph. From the graph find the quantity of Hydrogen if Oxygen is 12 grams and quantity of oxygen if hydrogen is $\frac{3}{2}$ gms.?

 [**Hint :** If the quantities of hydrogen and oxygen are 'x' and 'y' respectively, then $x : y = 1:8 \Rightarrow 8x = y$]

8. In a mixture of 28 litres, the ratio of milk and water is 5:2. Set up the equation between the mixture and milk. Draw its graph. By observing the graph find the quantity of milk in the mixture.

 [**Hint:** Ratio between mixture and milk $= 5 + 2 : 5 = 7 : 5$]

9. In countries like USA and Canada temperature is measured in Fahrenheit where as in countries like India, it is measured in Celsius. Here is a linear equation that converts Fahrenheit to Celsius

 $F = \left(\frac{9}{5}\right)C + 32$

 (i) Draw the graph of the above linear equation having Celsius on x-axis and Fahrenheit on Y-axis.

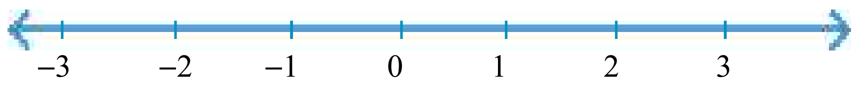
 (ii) If the temperature is 30°C, what is the temperature in Fahrenheit?

 (iii) If the temperature is 95°F, what is the temperature in Celsius?

 (iv) Is there a temperature that has numerically the same value in both Fahrenheit and Celsius? If yes find it?

6.5 EQUATION OF LINES PARALLEL TO X-AXIS AND Y-AXIS

Consider the equation $x = 3$. If this is treated as an equation in one variable x, then it has the unique solution $x = 3$ which is a point on the number line



However when treated as an equation in two variables and plotted on the coordinate plane it can be expressed as $x + 0.y - 3 = 0$. This has infinitely many solutions, let us find some of them.

Here the coefficient of y is zero. So for all values of y, x becomes 3.

Table of solutions

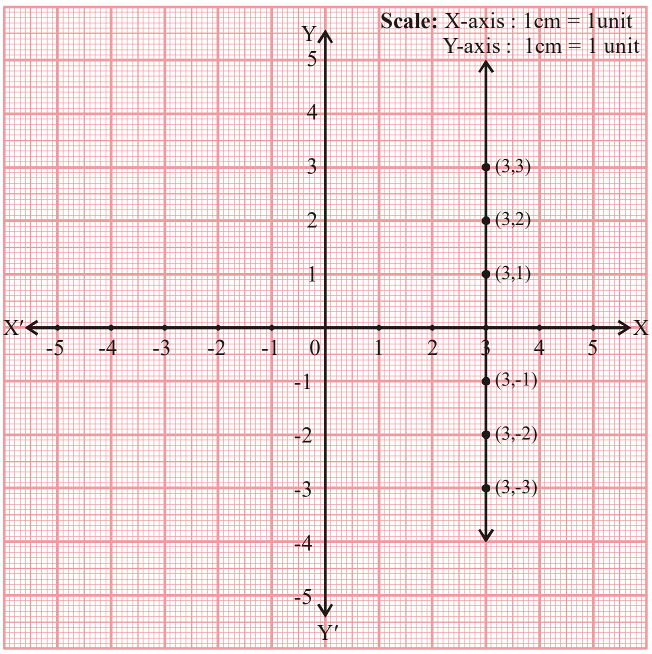
x	3	3	3	3	3	3	……
y	1	2	3	–1	–2	–3	…..
(x, y)	(3, 1)	(3, 2)	(3, 3)	(3, –1)	(3, –2)	(3, –3)	…..
Points	A	B	C	D	E	F	…..

From the table it is clear that this equation has infinitely many solutions of the form (3, a) where a is any real number.

Now draw the graph using the above solutions. What do you notice from the graph?

Is it a straight line? Whether it is any line or axes? The line drawn is a straight line and is parallel to Y-axis?

What is the distance of this line from the y-axis?



Thus the graph of $x = 3$ is a line parallel to the y-axis at a distance of 3 units to the right of it.

Do This

1. i) Draw the graph of following equations.

   a) $x = 2$ b) $x = -2$ c) $x = 4$ d) $x = -4$

   ii) Are the graphs of all these equations parallel to Y-axis?

   iii) Find the distance between the lines in the graph and the Y-axis in each case.

2. i) Draw the graph of the following equations

   a) $y = 2$ b) $y = -2$ c) $y = 3$ d) $y = -3$

   ii) Are all these parallel to the X-axis?

   iii) Find the distance between the graph of the line and the X-axis in each case

From the above observations we can conclude the following:

1. The graph of $x = k$ is a line parallel to Y-axis at a distance of k units and passing through the point $(k, 0)$
2. The graph of $y = k$ is a line parallel to X-axis at a distance of k units and passing through the point $(0, k)$

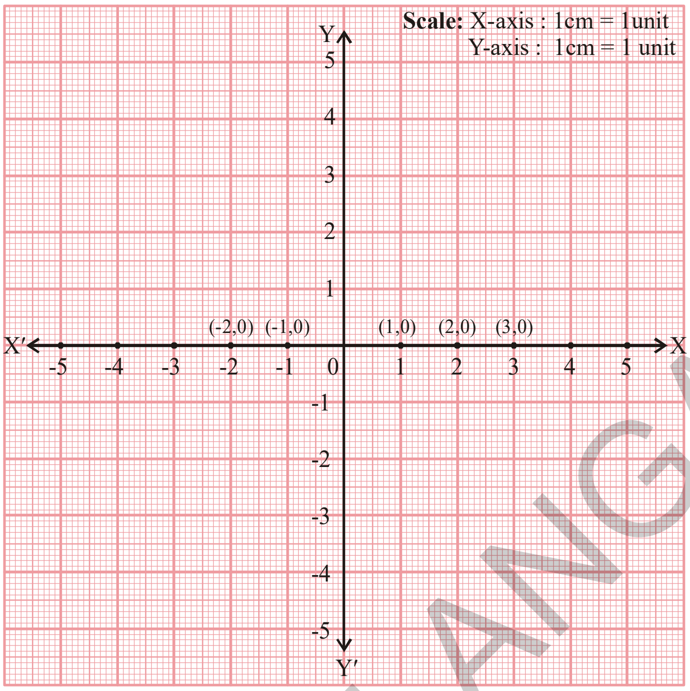
5.5.1 Equation of the X-axis and the Y-axis:

Consider the equation $y = 0$. It can be written as $0.x + y = 0$. Let us draw the graph of this equation.

Table of solutions

x	1	2	3	–1	–2	…...
y	0	0	0	0	0	…..
(x, y)	(1, 0)	(2, 0)	(3, 0)	(–1, 0)	(–2, 0)	…..
Points	A	B	C	D	E	…..

By plotting all these points on the graph paper, we get the following figure. From the graph what do you notice?



We notice that all these points lie on the X-axis and y-coordinate of all these points is '0'.

Therefore the equation $y = 0$ represents X-axis. In other words the equation of the X-axis is $y = 0$.

Try This

Find the equation of Y-axis.

Exercise - 6.5

1. Give the graphical representation of the following equation.

   a) On the number line and b) On the Cartesian plane

   i) $x = 3$ ii) $y + 3 = 0$ iii) $y = 4$ iv) $2x - 9 = 0$

   v) $3x + 5 = 0$

2. Give the graphical representation of $2x - 11 = 0$ as an equation in

   i) one variable ii) two variables

3. Solve the equation $3x + 2 = 8x - 8$ and represent the solution on

   i) the number line   ii) the Cartesian plane

4. Write the equation of the line parallel to X-axis, and passing through the point

   i) $(0, -3)$ ii) $(0, 4)$ iii) $(2, -5)$ iv) $(3, 4)$

5. Write the equation of the line parallel to Y-axis and passing through the point

   i) $(-4, 0)$ ii) $(2, 0)$ iii) $(3, 5)$ iv) $(-4, -3)$

6. Write the equation of three lines that are

   (i) parallel to the X-axis   (ii) parallel to the Y-axis.

What we have discussed ?

1. If a linear equation has two variables then it is called linear equation in two variables.
2. Any pair of values of 'x' and 'y' which satisfy the linear equation in two variables is called its solution.
3. A linear equation in the two variables has many solutions.
4. The graph of every linear equation in two variables is a straight line.
5. An equation of the form $y = mx$ represents a line passes through the origin.
6. The graph of $x = k$ is a line parallel to Y - axis at a distance of k units and passes through the point (k, 0).
7. The graph of $y = k$ is a line parallel to X-axis at a distance of k units and passes through the point (0, k).
8. Equation of X-axis is $y = 0$.
9. Equation of Y-axis is $x = 0$.

B9E6E6

Chapter

7

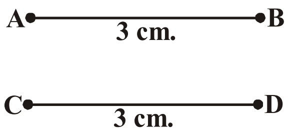
Triangles

P4G7T4

7.1 Introduction

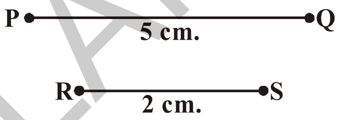
We have drawn figures with lines and curves and studied their properties. Do you remember, how to draw a line segment of a given length? All line segments are not same in size, they can be of different lengths. We also draw circles. What measure do we need and have been used to draw a circle? We need the radius of the circle. We also draw angles equal to the given angle.

We know if the lengths of two line segments are equal then they are congruent.



$\overline{AB} \cong \overline{CD}$

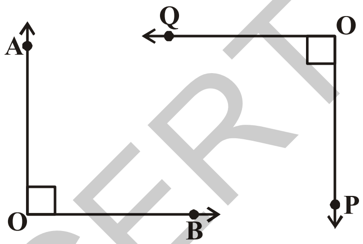
(Congruent)



$\overline{PQ} \ncong \overline{RS}$

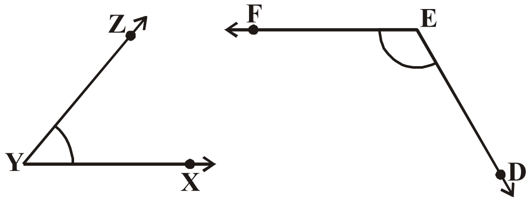
(Non-congruent)

Two angles are congruent, if their angle measure is same.



$\angle AOB \cong \angle POQ$

(Congruent)



$\angle XYZ \ncong \angle DEF$

(Non-congruent)

From the above examples we can say that to make or check whether the figures are same in size or not we need some specific information about the measures describing these figures.

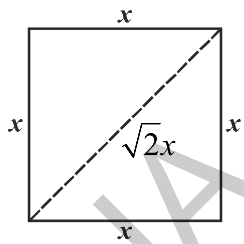
Let's consider a square : What is the minimum information required to say whether two squares are of the same size or not?

Satya said- "I only need the measure of the side of the given squares". If the sides of given squares are equal then the squares are of identical size.

Siri said "that is right but even if the diagonals of the two squares are equal then we can say that the given squares are identical and are same in size".

Do you think both of them are right?

Recall the properties of a square. You can't make two different squares with sides having same measures. Can you? And the diagonals of two squares can only be equal when their sides are equal. See the given figure:

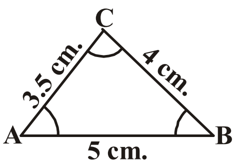


The figures that are same in shape and size are called congruent figures ('Congruent' means equal in all aspects). Hence squares that have sides with same measure are congruent and also with equal diagonals are congruent.

Note : In general, sides decide sizes and angles decide shapes.

We know if two squares are congruent and we trace one out of them on a paper and place it on other one, it will cover the other exactly.

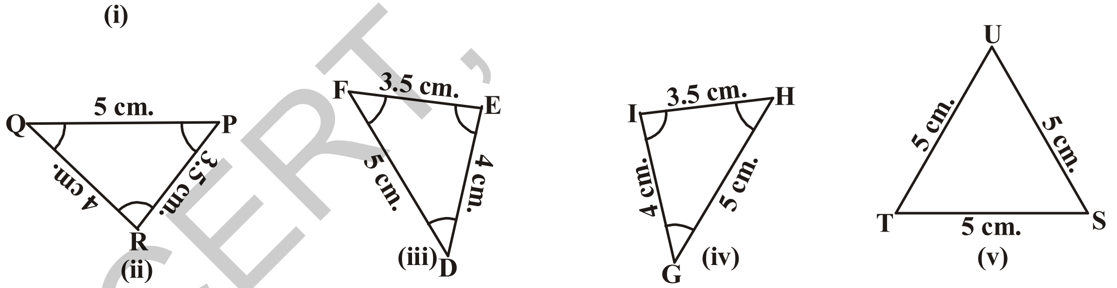
Then we can say that sides, angles, diagonals of one square are respectively congruent to the sides, angles and diagonals of the other square. Let us now consider the congruence of two triangles.



(i)

We know that if two traingles are congruent then the sides and angles of one triangle are congruent to the corresponding sides and angles of the other triangle.

Which of the triangles given below are congruent to triangle ABC in fig.(i).



If we trace these triangles from fig.(ii) to (v) and try to cover ΔABC. We would observe that triangles in fig.(ii), (iii) and (iv) are congruent to ΔABC while ΔTSU in fig.(v) is not congruent to ΔABC.

If ΔPQR is congruent to ΔABC, we write it as ΔPQR $\cong$ ΔABC.

Notice that when ΔPQR $\cong$ ΔABC, then sides of ΔPQR covers the corresponding sides of ΔABC equally and so do the angles.

That is, PQ coincides AB, QR coincides BC and RP coincides CA; $\angle$P coincides $\angle$A, $\angle$Q coincides $\angle$B and $\angle$R coincides $\angle$C. Also, there is a one-one correspondence between the vertices. That is, P corresponds to A, Q to B, R to C.

We write as symbolically. $P \leftrightarrow A$, $Q \leftrightarrow B$, $R \leftrightarrow C$ $\quad \Delta PQR \cong \Delta ABC$

Note that under order of correspondence, $\Delta PQR \cong \Delta ABC$; but it will not be correct to write $\Delta QRP \cong \Delta ABC$, $Q \leftrightarrow A$; $R \leftrightarrow B$; $P \leftrightarrow C$ and as we get $QR = AB$, $RP = BC$ and $QP = AC$ which is incorrect for the given figures?

Similarly, for fig. (iii),

$FD \leftrightarrow AB$, $DE \leftrightarrow BC$ and $EF \leftrightarrow CA$

and $F \leftrightarrow A$, $D \leftrightarrow B$ and $E \leftrightarrow C$

So, $\Delta FDE \cong \Delta ABC$ but writing $\Delta DEF \cong \Delta ABC$ is not correct.

In the same way try to write the congruency for ΔABC with fig.(iv)

So, it is necessary to write the correspondence of vertices in correct order, while writing of congruence of triangles.

Note that **corresponding parts of congruent triangles** are **equal** and we write in short as 'CPCT' for *corresponding parts of congruent triangles*.

Do This

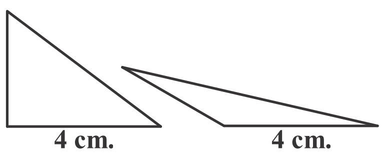
1. There are some statements given below. Write whether they are true or false :
   - i. Two circles are always congruent. ( )
   - ii. Two line segments of same length are always congruent. ( )
   - iii. Two right angle triangles are sometimes congruent. ( )
   - iv. Two equilateral triangles with their sides equal are always congruent. ( )
2. Howmany number of minimum measurements do you require to check if the given figures are congruent or not?
   - i. Two rectangles
   - ii. Two rhombuses

7.2 Criteria for Congruence of Triangles

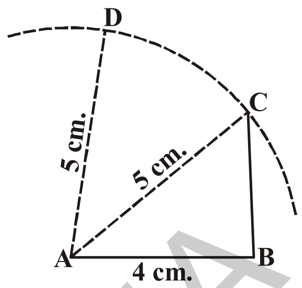
You have learnt the criteria for congruency of triangles in your earlier class. Let us recall.

Is it necessary to know all the three sides and three angles of a triangle to construct a unique triangle? Can we construct different triangles with the same given measurements?

Draw two triangles with one of its sides 4 cm. Can you make two different triangles with one side of 4 cm? Discuss with your friends. Do you all get congruent triangles? You can draw different types of triangles if one side is given say 4 cm.

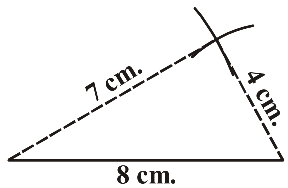


Now take two sides as 4 cm. and 5 cm. and draw as many triangles as you can. Do you get congruent triangles?



We can make different triangles even with these two given measurements.

Now draw triangles with sides 4 cm., 7 cm. and 8 cm.

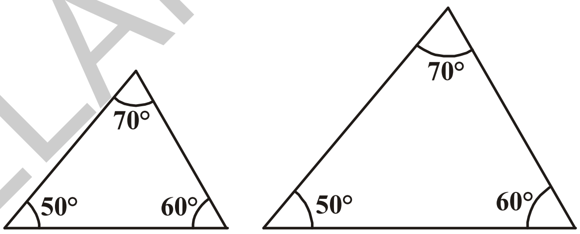


Can you draw two different triangles? You find that these three with measurement sides, we will have a unique triangle. If at all you draw the triangles with these measures they will be congruent to this unique triangle.

Now take three angles of your choice, of course the sum of the angles must be 180°. Draw two triangles for your chosen angle measurement.

Mahima finds that she can make different triangles by using three angle measurement.

$\angle A = 50°$, $\angle B = 70°$, $\angle C = 60°$



So it seems that knowing the 3 angles is not enough to make a specific triangle.

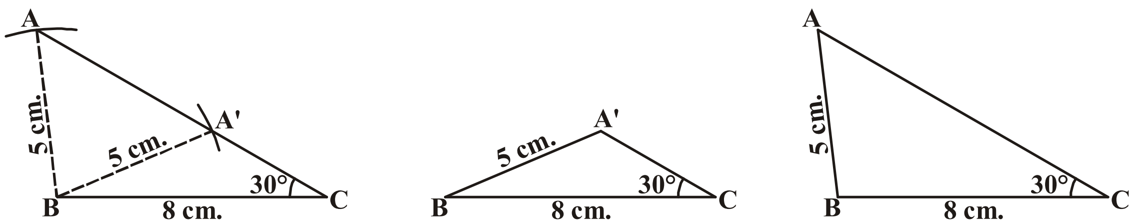
Sharif thought that if two angles are given then he could easily find the third one by using the angle sum property of a triangle. So measures of two angles is enough to draw a triangle but not uniquely. Hence knowing 3 or 2 angles is not adequate. We need at least three specific and independent measurements (elements) to make a unique triangle.

Now try to draw two distinct triangles with each sets of these three measurements:

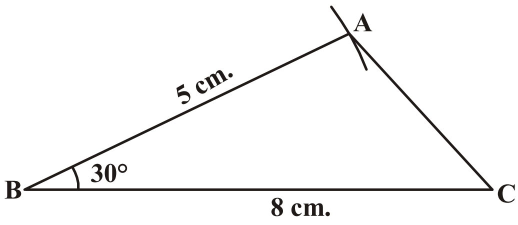
i. ΔABC where $AB = 5$ cm., $BC = 8$ cm., $\angle C = 30°$

ii. ΔABC where $AB = 5$ cm., $BC = 8$ cm., $\angle B = 30°$

(i) Are you able to draw a unique triangle with the given measurements, draw and check with your friends.



Here we can draw two different triangles ΔABC and $\Delta A'BC$ with given measurements. Now draw two triangles with given measurements (ii). What do you observe? They are congruent triangles. Aren't they?



In the other words you can draw a unique triangle with the measurements given in case(ii).

Have you noticed the order of measures given in case (i) & case (ii)? In case (i) two sides and one angle are given which is not an included angle but in case (ii) included angle is given along with two sides. Thus given two sides and one angle i.e. three independent measures is not the only criteria to make a unique triangle. But the order of given measurements to construct a triangle also plays a vital role in constructing a unique triangle.

7.3 Congruence of Triangles

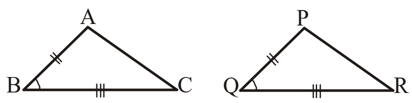
The above has an implication for checking the congruency of triangles. If we have two triangles with one side equal or two triangles with all 3 angles equal, we can not conclude that triangles are congruent as there are more than one triangle possible with these specifications. Even when we have two sides and an angle equal we cannot say that the triangles are congruent unless the angle is between the given sides. We can say that the SAS (side angle side) congruency rule holds but not SSA or ASS.

We take this as the first criterion for congruency of triangles and prove the other criteria through this.

Axiom (SAS congruence rule): *Two triangles are congruent if two sides and the included angle of one triangle are equal to the two sides and the included angle of the other triangle.*

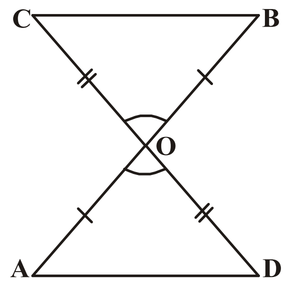
For example. In ΔABC and ΔPQR

AB=PQ, BC=QR and $\angle ABC = \angle PQR$, then $\Delta ABC \cong \Delta PQR$



Example-1. In the given figure AB and CD are intersecting at 'O', OA = OB and OD = OC. Show that

(i) $\Delta AOD \cong \Delta BOC$ and (ii) AD || BC.



Solution : (i) In ΔAOD and ΔBOC,

OA = OB (given)

OD = OC (given)

Also, since $\angle AOD$ and $\angle BOC$ form a pair of vertically opposite angles, we have $\angle AOD = \angle BOC$.

So, $\Delta AOD \cong \Delta BOC$ (by the SAS congruence rule)

(ii) In congruent triangles AOD and BOC, the other corresponding parts are also equal.

So, $\angle OAD = \angle OBC$ and these form a pair of alternate angles for line segments AD and BC. (and transversal AB)

Therefore AD || BC

Example-2. AB is a line segment and line l is its perpendicular bisector. If a point P lies on l, show that P is equidistant from A and B.

Solution : Line $l \perp$ AB and it passes through C which is the mid-point of AB

We have to show that PA = PB.

Consider ΔPCA and ΔPCB.

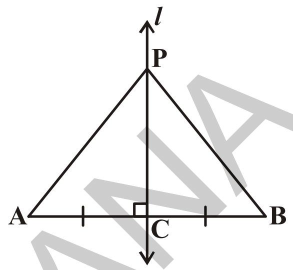
We have AC = BC (C is the mid-point of AB)

$\angle$PCA = $\angle$PCB = 90° (Given)

PC = PC (Common side)

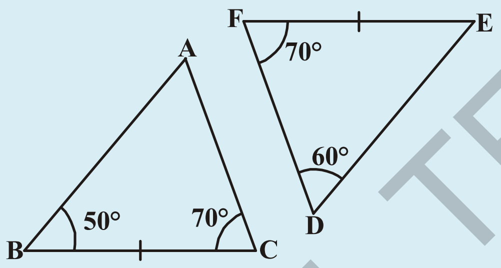
So, $\Delta PCA \cong \Delta PCB$ (SAS rule)

and so, PA = PB, (corresponding sides of congruent triangles.)

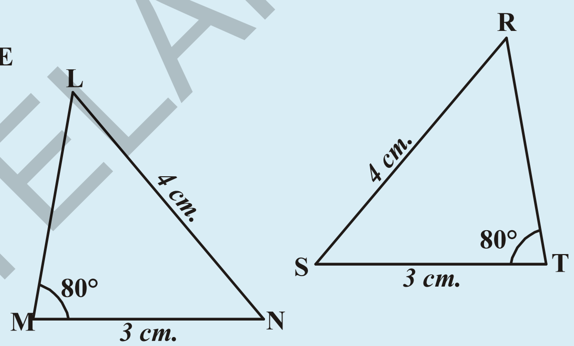


Do These

1. State whether the following triangles are congruent or not? Give reasons for your answer.

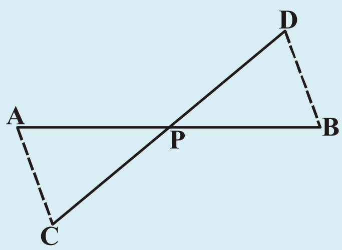


(i)



(ii)

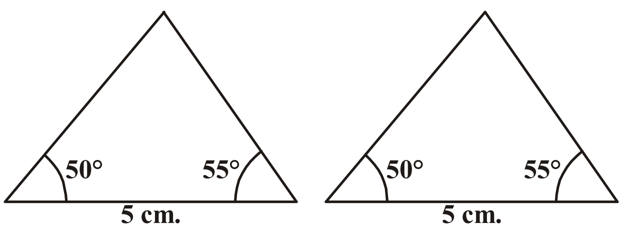
2. In the given figure, the point P bisects AB and DC. Prove that $\Delta APC \cong \Delta BPD$



7.3.1 Other Congruence Rules

Try to construct two triangles in which two of the angles are 50° and 55° and the side on which both these angles lie being 5cm.

Cut out these triangles and place one on the other. What do you observe? You will find that both the triangles are congruent. This result is the angle-side-angle criterion for congruence and is written as ASA criterion you have seen



this in earlier classes. Now let us state and prove the result. Since this result can be proved, it is called a theorem and to prove it, we use the SAS axiom for congruence.

Theorem 7.1 (ASA congruence rule) : Two triangles are congruent, if two angles and the included side of one triangle are equal to two angles and the included side of the other triangle.

Given: In ΔABC and ΔDEF

$\angle B = \angle E$, $\angle C = \angle F$ and $BC = EF$

Required To Prove (RTP): $\Delta ABC \cong \Delta DEF$

Proof: There will be three posibilities. The possiblities between AB and DE are either $AB > DE$ or $AB < DE$ or $AB = DE$.

We will consider all these cases and see what does it mean for ΔABC and ΔDEF?

Case (i): Let $AB = DE$, Now what do we observe?

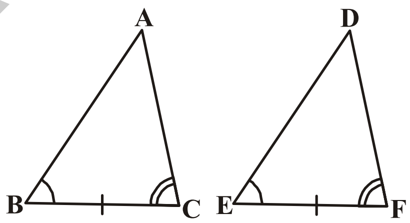
Consider ΔABC and ΔDEF

$AB = DE$ (Assumed)

$\angle B = \angle E$ (Given)

$BC = EF$ (Given)

So, $\Delta ABC \cong \Delta DEF$ (By SAS congruency axiom)



Case (ii): Let the second case be $AB > DE$.

So, we can take a point P on AB such that $PB = DE$.

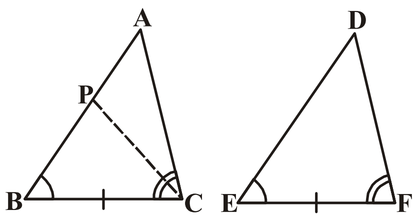
Now consider ΔPBC and ΔDEF

$PB = DE$ (by construction)

$\angle B = \angle E$ (given)

$BC = EF$ (given)

So, $\Delta PBC \cong \Delta DEF$ (by SAS congruency axiom)



Since the triangles are congruent their corresponding parts are equal.

So, $\angle PCB = \angle DFE$

But, $\angle ACB = \angle DFE$ (given)

So $\angle ACB = \angle PCB$ (from the above)

Is this possible?

This is possible only if P coincides with A

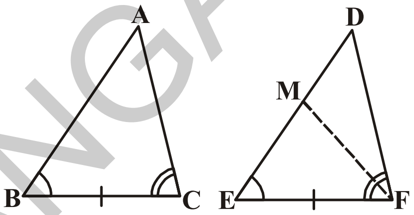
(or) $BA = ED$

So, $\Delta ABC \cong \Delta DEF$ (By SAS congruency axiom)

(**Note :** We have shown above that if $\angle B = \angle E$ and $\angle C = \angle F$ and $BC = EF$ then $AB = DE$ and the two triangles are congruent by SAS rule).

Case (iii): Let the third case be $AB < DE$

We can choose a point M on DE such that $ME = AB$ and repeating the arguments as given in case (ii), we can conclude that $AB = DE$ and so, $\Delta ABC \cong \Delta DEF$. Look at the figure and try to prove it yourself.



Suppose, now in two triangles two pairs of angles and one pair of corresponding sides are equal but the side is not included between the corresponding equal pairs of angles. Are the triangles still congruent? You will observe that they are congruent. Can you reason out why?

You know that the sum of the three angles of a triangle is 180°. So if two pairs of angles are equal, the third pair is also equal (180° – sum of equal angles).

So, two *triangles are congruent if any two pairs of angles and one pair of corresponding sides are equal.* We may call it as the **AAS Congruence Rule.** Let us examine some more examples.

Example-3. In the given figure, $AB \parallel DC$ and $AD \parallel BC$ show that $\Delta ABC \cong \Delta CDA$

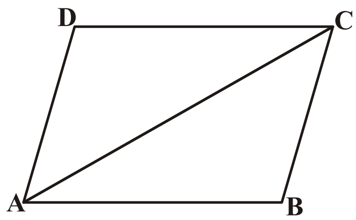
Solution : Consider ΔABC and ΔCDA

$\angle BAC = \angle DCA$ (alternate interior angles)

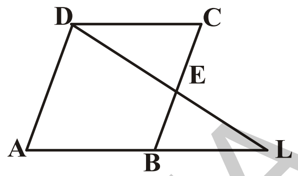
$AC = CA$ (common side)

$\angle BCA = \angle DAC$ (alternate interior angles)

$\Delta ABC \cong \Delta CDA$ (by ASA congruency)



Example-4. In the given figure, $AL \parallel DC$, E is mid point of BC. Show that $\Delta EBL \cong \Delta ECD$



Solution : Consider ΔEBL and ΔECD

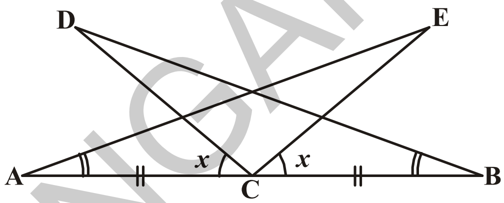
$\angle BEL = \angle CED$ (vertically opposite angles)

$BE = CE$ (since E is mid point of BC)

$\angle EBL = \angle ECD$ (alternate interior angles)

$\Delta EBL \cong \Delta ECD$ (by ASA congruency)

Example-5. Use the information given in the adjoining figure, prove the following :



(i) $\Delta DBC \cong \Delta EAC$

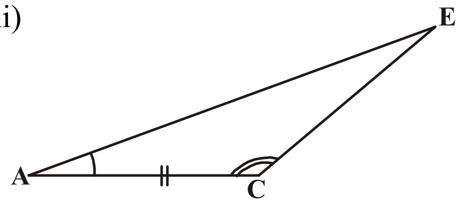
(ii) $DC = EC$.

Solution : Let $\angle ACD = \angle BCE = x$

$\therefore \angle ACE = \angle DCE + \angle ACD = \angle DCE + x$ ...... (i)

$\therefore \angle BCD = \angle DCE + \angle BCE = \angle DCE + x$ ...... (ii)

From (i) and (ii), we get : $\angle ACE = \angle BCD$

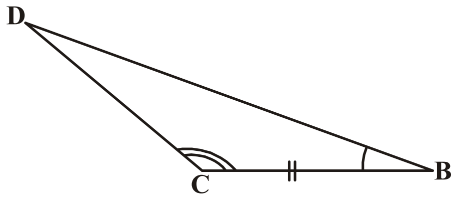


Now in ΔDBC and ΔEAC,

$\angle ACE = \angle BCD$ (proved above)

$BC = AC$ [Given]

$\angle CBD = \angle CAE$ [Given]

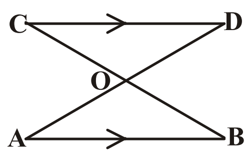


$\Delta DBC \cong \Delta EAC$ [By A.S.A]

since $\Delta DBC \cong \Delta EAC$

$DC = EC$ (by CPCT)

Example-6: Line-segment AB line-segment CD are parallel. O is the mid-point of AD. Show that (i) $\Delta AOB \cong \Delta DOC$ (ii) O is also the mid-point of BC.



Solution : (i) Consider ΔAOB and ΔDOC.

$\angle ABO = \angle DCO$ (Alternate angles as $AB \parallel CD$ and BC is the transversal)

$\angle AOB = \angle DOC$ (Vertically opposite angles)

$OA = OD$ (Given)

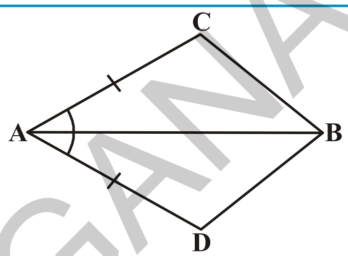
Therefore, $\Delta AOB \cong \Delta DOC$ (AAS rule)

(ii) $OB = OC$ (CPCT)

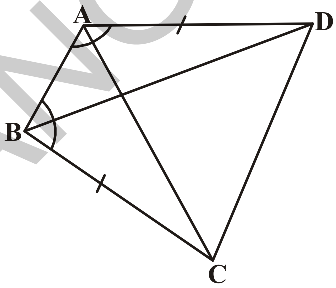
$\therefore$, O is the mid-point of BC.

EXERCISE - 7.1

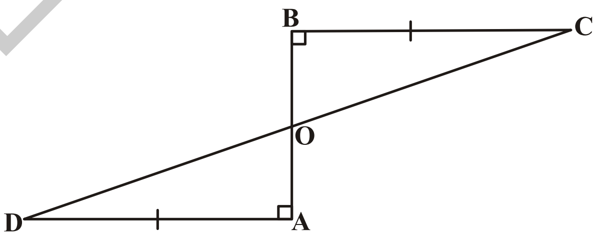
1. In quadrilateral ACBD, $AC = AD$ and AB bisects $\angle A$. Show that $\Delta ABC \cong \Delta ABD$.
What can you say about BC and BD?



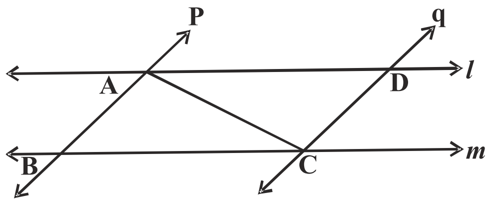
2. ABCD is a quadrilateral in which $AD = BC$ and $\angle DAB = \angle CBA$ Prove that
(i) $\Delta ABD \cong \Delta BAC$
(ii) $BD = AC$
(iii) $\angle ABD = \angle BAC$



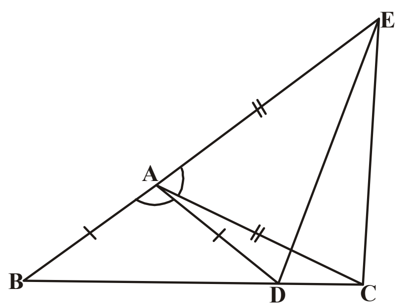
3. AD and BC are equal and perpendiculars to a line segment AB then show that CD bisects AB.



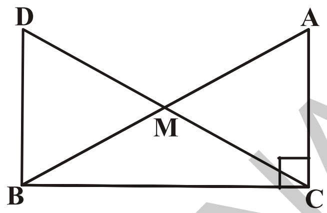
4. l and m are two parallel lines intersected by another pair of parallel lines p and q . Show that $\Delta ABC \cong \Delta CDA$



5. In the adjacent figure, $AC = AE$, $AB = AD$ and $\angle BAD = \angle EAC$. Show that $BC = DE$.



6. In right triangle ABC, right angle is at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that DM = CM. Point D is joined to point B (see figure). Show that :



(i) $\Delta AMC \cong \Delta BMD$

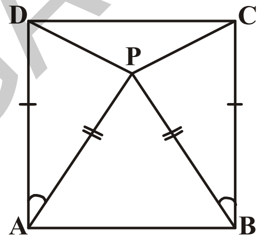
(ii) $\angle DBC$ is a right angle

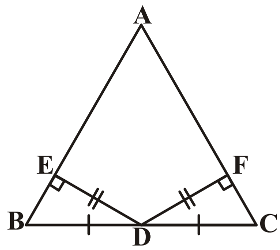
(iii) $\Delta DBC \cong \Delta ACB$

(iv) $CM = \frac{1}{2} AB$

7. In the adjacent figure ABCD is a square and ΔAPB is an equilateral triangle. Prove that $\Delta APD \cong \Delta BPC$.

(**Hint :** In ΔAPD and ΔBPC, $AD = BC$, $AP = BP$ and $\angle PAD = \angle PBC = 90^o - 60^o = 30^o$]



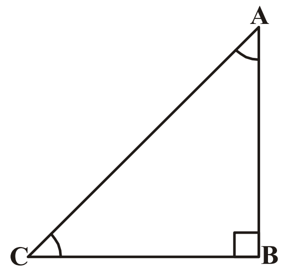


8. In the adjacent figure ΔABC, D is the midpoint of BC. $DE \perp AB$, $DF \perp AC$ and $DE = DF$. Show that $\Delta BED \cong \Delta CFD$.

9. If the bisector of an angle of a triangle also bisects the opposite side, prove that the triangle is isosceles.

10. In the given figure ABC is a right triangle and right angled at B such that $\angle BCA = 2\angle BAC$. Show that hypotenuse $AC = 2BC$.

(**Hint** : Produce CB to a point D that BC = BD)

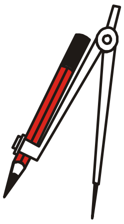


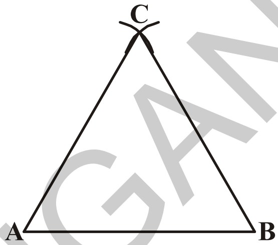
7.4 Some properties of a triangle

In the above section you have studied two criteria for the congruence of triangles. Let us now apply these results to study some properties related to a triangle whose two sides are equal.

ACTIVITY

i. To contruct a triangle using compass, take any measurement and draw a line segment AB. Now open a compass with sufficient length and put it on point A and B and draw an arc. Which type of triangle do you get? Yes this is an isosceles triangle. So, ΔABC in figure is an isosceles triangle with $AC = BC$. Now measure $\angle A$ and $\angle B$. What do you observe?





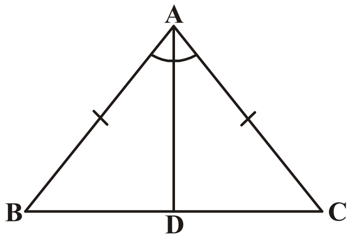
ii. Cut an isosceles triangle

Now fold the triangle so that two congruent sides fit precisely one on top of the other. What do you notice about $\angle A$ and $\angle B$?

You may observe that in each such triangle, the angles opposite to the equal sides are equal.

This is a very important result and is indeed true for any isosceles triangle. It can be proved as shown below.

Theorem-7.2 : Angles opposite to equal sides of an isosceles triangle are equal.



This result can be proved in many ways. One of the proofs is given here.

Given: ΔABC is an isosceles triangle in which $AB = AC$.

RTP: $\angle B = \angle C$.

Construction: Let us draw the bisector of $\angle A$ and let D be the point of intersection of this bisector of $\angle A$ and BC.

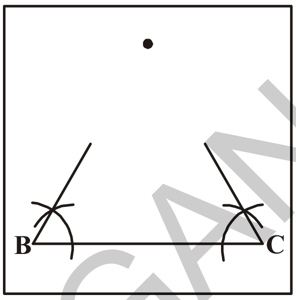
Proof : In ΔBAD and ΔCAD,

$AB = AC$	(Given)
$\angle BAD = \angle CAD$	(By construction)
$AD = AD$	(Common side)
$\therefore \Delta BAD \cong \Delta CAD$	(By SAS congruency axiom)
So, $\angle ABD = \angle ACD$	(By CPCT)
i.e., $\angle B = \angle C$	(Same angles)

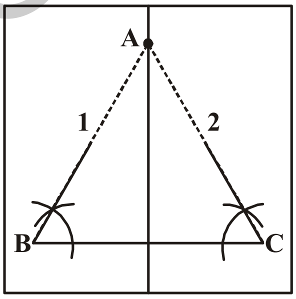
Is the converse also true? That is "If two angles of any triangle are equal, can we say that the sides opposite to them are also equal?"

Activity

1. On a tracing paper draw a line segment BC of length 6cm.
2. From vertices B and C draw rays with angle 60^0 each. Name the point A where they meet.
3. Fold the paper so that B and C fit precisely on top of each other. What do you observe? Is AB = AC?



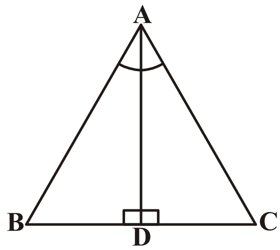
Repeat this activity by taking different angles for $\angle B$ and $\angle C$. Each time you will observe that the sides opposite to equal angles are equal. So we have the following



Theorem-7.3 : The sides opposite to equal angles of a triangle are equal.

This is the converse of previous Theorem. You are advised to prove this using ASA congruence rule.

Example-7. In ΔABC, the bisector AD of $\angle A$ is perpendicular to side BC Show that AB = AC and ΔABC is isosceles.



Solution : In ΔABD and ΔACD,

$\angle BAD = \angle CAD$ (Given)

AD = AD (Common side)

$\angle ADB = \angle ADC = 90°$ (Given)

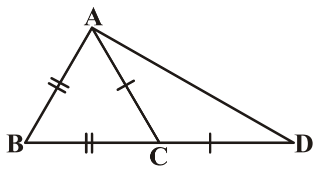
$\therefore \Delta ABD \cong \Delta ACD$ (ASA rule)

So, AB = AC (CPCT)

or, ΔABC is an isosceles triangle.

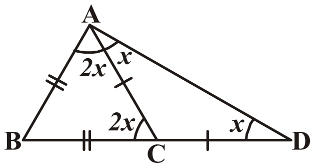
Example-8. In the adjacent figure, AB = BC and AC = CD.

Prove that : $\angle BAD : \angle ADB = 3 : 1$.



Solution : Let $\angle ADB = x$

In ΔACD, $AC = CD$



$\Rightarrow$ $\angle CAD = \angle CDA = x$

and, ext. $\angle ACB = \angle CAD + \angle CDA$

$= x + x = 2x$

$\Rightarrow$ $\angle BAC = \angle ACB = 2x$. ($\because$ In ΔABC, $AB = BC$)

$\therefore$ $\angle BAD = \angle BAC + \angle CAD$

$= 2x + x = 3x$

And, $\dfrac{\angle BAD}{\angle ADB} = \dfrac{3x}{x} = \dfrac{3}{1}$

i.e., $\angle BAD : \angle ADB = 3 : 1.$

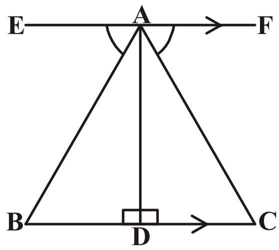
Hence Proved.

Example-9. In the given figure, AD is perpendicular to BC and EF || BC, if $\angle EAB = \angle FAC$, show that triangles ABD and ACD are congruent.

Also, find the values of x and y if $AB = 2x + 3$, $AC = 3y + 1$,

$BD = x$ and $DC = y + 1$.

Solution : $AD \perp EF$



$\Rightarrow$ $\angle EAD = \angle FAD = 90^0$

$\angle EAB = \angle FAC$ (given)

$\Rightarrow$ $\angle EAD - \angle EAB = \angle FAD - \angle FAC$

$\Rightarrow$ $\angle BAD = \angle CAD$

In ΔABD and ΔACD

$\angle BAD = \angle CAD$ [proved above]

$\angle ADB = \angle ADC = 90^0$ [Given $AD \perp BC$]

and $AD = AD$ (Common side)

$\therefore$ $\Delta ABD \cong \Delta ACD$ [By ASA]

Hence proved.

$\angle ABD = \angle ACD$ $\Rightarrow AB = AC$ and $BD = CD$ [By C.P.C.T]

$\Rightarrow 2x + 3 = 3y + 1$ and $x = y + 1$

$\Rightarrow 2x - 3y = -2$ and $x - y = 1$

Substituting $x = 1 + y$: $2(1+y) - 3y = -2$

$2 + 2y - 3y = -2$

$-y = -2 - 2$

$-y = -4$

Substituting $y = 4$ in $x = 1 + y$

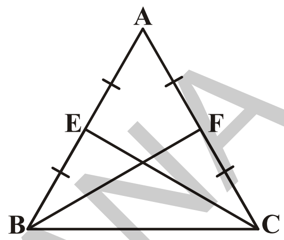
$x = 1 + 4$

$x = 5$

$\therefore$ $x = 5,\ y = 4$

Example-10. E and F are respectively the mid-points of equal sides AB and AC of ΔABC (see figure) Show that BF = CE

Solution : In ΔABF and ΔACE,



AB = AC (Given)

$\angle A = \angle A$ (common angle)

AF = AE (Halves of equal sides)

$\therefore$ $\Delta ABF \cong \Delta ACE$ (SAS rule)

So, BF = CE (CPCT)

Example-11. In an isosceles triangle ABC with AB = AC, D and E are points on BC such that BE = CD (see figure) Show that AD = AE,

Solution : In ΔABD and ΔACE,

AB = AC (given) .... (1)

$\angle B = \angle C$ (Angles opposite to equal sides) .... (2)

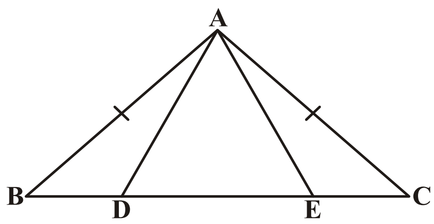
Also, BE = CD

So, BE – DE = CD – DE

That is, BD = CE (3)

So, $\Delta ABD \cong \Delta ACE$

(Using (1), (2), (3) and SAS rule).

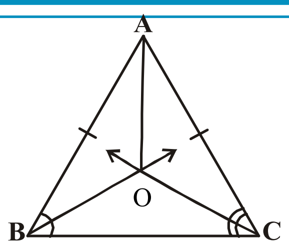


This gives AD = AE (CPCT)

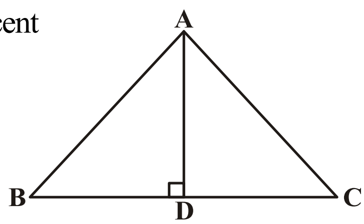
EXERCISE - 7.2

1. In an isosceles triangle ABC, with AB = AC, the bisectors of $\angle B$ and $\angle C$ intersect each other at O. Join A to O. Show that :

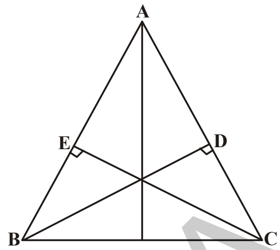
   (i) OB = OC (ii) AO bisects $\angle A$

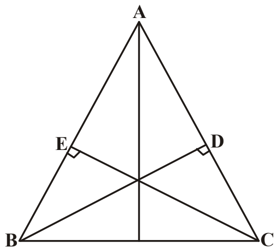


2. In ΔABC, AD is the perpendicular bisector of BC (See adjacent figure). Show that ΔABC is an isosceles triangle in which AB = AC.



3. ABC is an isosceles triangle in which altitudes BD and CE are drawn to equal sides AC and AB respectively (see figure) Show that these altitudes are equal.

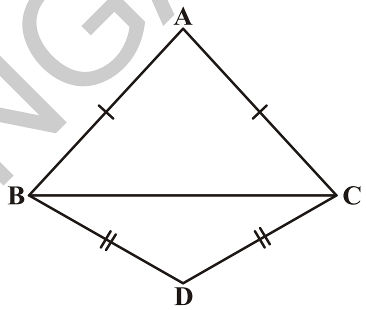




4. ABC is a triangle in which altitudes BD and CE to sides AC and AB are equal (see figure) . Show that

(i) $\Delta ABD \cong \Delta ACE$

(ii) AB = AC i.e., ABC is an isosceles triangle.



5. ΔABC and ΔDBC are two isosceles triangles on the same base BC (see figure). Show that $\angle ABD = \angle ACD$.

7.5 Some more criteria for congruency of triangles

Theorem 7.4 (SSS congruence rule) : Through construction we have seen that SSS congruency rule hold. This theorem can be proved using a suitable construction.

In two triangles, if the three sides of one triangle are respectively equal to the corresponding three sides of another triangle, then the two triangles are congruent.

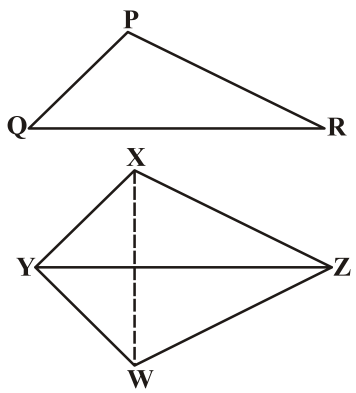
• **Proof for SSS Congruence Rule**

Given: ΔPQR and ΔXYZ are such that PQ = XY, QR = YZ and PR = XZ

To Prove : $\Delta PQR \cong \Delta XYZ$

Construction : Draw YW such that $\angle ZYW = \angle PQR$ and WY = PQ. Join XW and WZ

Proof: In ΔPQR and ΔWYZ



QR = YZ	(Given)
$\angle PQR = \angle ZYW$	(Construction)
PQ = YW	(Construction)
$\therefore \Delta PQR \cong \Delta WYZ$	(SAS congruence axiom)

$\Rightarrow \angle P = \angle W$ and $PR = WZ$ (CPCT)

$PQ = XY$ (given) and $PQ = YW$ (Construction)

$\therefore XY = YW$

Similarly, $XZ = WZ$

In ΔXYW, $XY = YW$

$\Rightarrow \angle YWX = \angle YXW$ (In a triangle, equal sides have equal angles opposite to them)

Similarly, $\angle ZWX = \angle ZXW$

$\therefore \angle YWX + \angle ZWX = \angle YXW + \angle ZXW$

$\Rightarrow \angle W = \angle X$

Now, $\angle W = \angle P$

$\therefore \angle P = \angle X$

In ΔPQR and ΔXYZ

$PQ = XY$

$\angle P = \angle X$

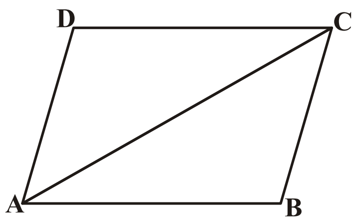
$PR = XZ$

$\therefore \Delta PQR \cong \Delta XYZ$ (SAS congruence criterion)

Let us see the following example based on it.

Example-12. In quadrilateral ABCD, $AB = CD$, $BC=AD$ show that $\Delta ABC \cong \Delta CDA$

Consider ΔABC and ΔCDA



Solution : $AB = CD$ (given)

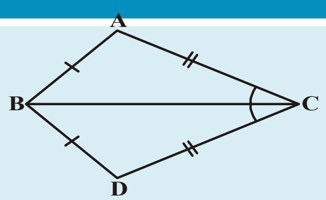
$AD = BC$ (given)

$AC = CA$ (common side)

$\Delta ABC \cong \Delta CDA$ (by SSS congruency rule)

Do This

1. In the adjacent figure ΔABC and ΔDBC are two triangles such that $AB = BD$ and $AC= CD$. Show that $\Delta ABC \cong \Delta DBC$.



You have already seen that in the SAS congruency axiom, the pair of equal angles has to be the included angle between the pairs of corresponding equal sides and if not so, two triangles may not be congruent.

ACTIVITY

Construct a right angled triangle with hypotenuse 5 cm. and one side 3 cm. long. How many different triangles can be constructed? Compare your triangle with those of the other members of your class. Are the triangles congruent? Cut them out and place one triangle over the other with equal side placed on each other. Turn the triangle if necessary what do you observe? You will find that two right triangles are congruent, if side and hypotenuse of one triangle are respectively equal to the correseponding side and hypotenuse of other triangle.

Note that the right angle is not the included angle in this case. So we arrive at the following congruency rule.

Theorem 7.5 (RHS congruence rule) : If in two right triangles the hypotenuse and one side of one triangle are equal to the hypotenuse and one side of the another triangle, then the two triangles are congruent.

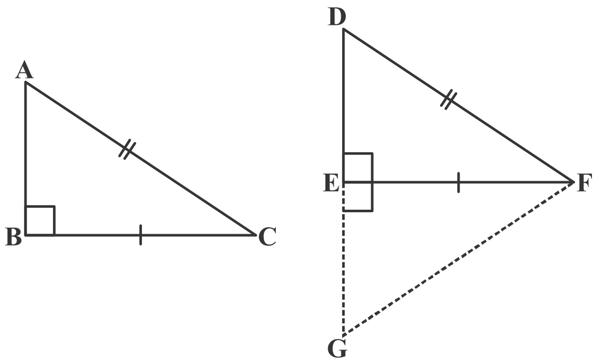
Note that RHS stands for right angle - hypotenuse-side.

Let us prove it.

Given: Two right triangles, ΔABC and ΔDEF

in which $\angle B = 90^\circ$ and $\angle E = 90^\circ$;

$AC = DF$ and $BC = EF$.



To prove: $\Delta ABC \cong \Delta DEF$

Construction: Produce DE to G

So that $EG = AB$. Join G, F.

Proof:

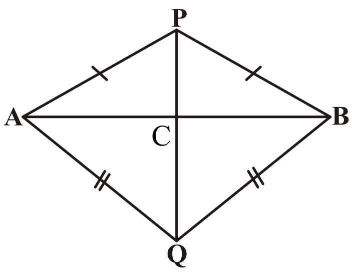
In ΔABC and ΔGEF

$AB = GE$	(By construction)
$\angle B = \angle FEG$	(Each angle is a right angle (90^0))
$BC = EF$	(Given)
$\Delta ABC \cong \Delta GEF$	(By SAS criterion of congruence)
So $\angle A = \angle G \ldots (1)$	(CPCT)

$AC = GF \ldots (2)$	(CPCT)
Further, $AC=GF$ and $AC=DF$	(From (2) and Given)
$\therefore\ DF = GF$	(From the above)
So, $\angle D = \angle G \ldots (3)$	(Angles opposite to equal sides are equal)
we get $\angle A = \angle D \ldots (4)$	(From (1) and (3))
Thus, in ΔABC and ΔDEF $\angle A = \angle D$,	(From (4))
$\angle B = \angle E$	(Given)
So, $\angle A + \angle B = \angle D + \angle E$	(on adding)
But $\angle A + \angle B + \angle C = 180^0$ and	(angle sum property of triangle)
$\angle D + \angle E + \angle F = 180^0$	
$180 - \angle C = 180 - \angle F$	($\angle A+\angle B=180^0-\angle C$ and $\angle D+\angle E=180^0-\angle F$)
So, $\angle C = \angle F, \ldots (5)$	(Cancellation laws)
Now, in ΔABC and ΔDEF, we have	
$BC = EF$	(given)
$\angle C = \angle F$	(from (5))
$AC = DF$	(given)
$\Delta ABC \cong \Delta DEF$	(by SAS axiom of congruence)

Example-13. AB is a linesegment. P and Q are points on either side of AB such that each of them is equidistant from the points A and B (See Fig). Show that the line PQ is the perpendicular bisector of AB.

Solution : You are given that $PA = PB$ and $QA = QB$ and you have to show that PQ is perpendicular on AB and PQ bisects AB. Let PQ intersect AB at C.



Can you think of two congruent triangles in this figure ?

Let us take ΔPAQ and ΔPBQ.

In these triangles,

$AP = BP$ (Given)

$AQ = BQ$ (Given)

$PQ = PQ$ (Common side)

So, $\Delta PAQ \cong \Delta PBQ$ (SSS rule)

Therefore, $\angle APQ = \angle BPQ$ (CPCT).

Now let us consider ΔPAC and ΔPBC.

You have : $AP = BP$ (Given)

$\angle APC = \angle BPC$ ($\angle APQ = \angle BPQ$ proved above)

$PC = PC$ (Common side)

So, $\Delta PAC \cong \Delta PBC$ (SAS rule)

$\therefore$ $AC = BC$ (CPCT) .... (1)

and $\angle ACP = \angle BCP$ (CPCT)

Also, $\angle ACP + \angle BCP = 180°$ (Linear pair)

So, $2\angle ACP = 180°$

or, $\angle ACP = 90°$ .... (2)

From (1) and (2), you can easily conclude that PQ is the perpendicular bisector of AB.

[Note that, without showing the congruence of ΔPAQ and ΔPBQ, we cannot show that $\Delta PAC \cong \Delta PBC$]

$AP = BP$ (Given)

$PC = PC$ (Common side)

and $\angle PAC = \angle PBC$ (Angles opposite to equal sides in ΔAPB)

It is because these results give us SSA rule which is not always valid or true for congruence of triangles as the given angle is not included between the equal pairs of sides.]

Now observe some more examples.

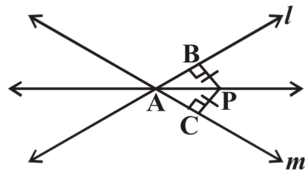
Example-14. P is a point equidistant from two lines l and m intersecting at point A (see figure). Show that the line AP bisects the angle between them.

Solution : You are given that lines l and m intersect each other at A.

$PB \perp l$ and $PC \perp m$. It is given that $PB = PC$.

You need to show that $\angle PAB = \angle PAC$.

Let us consider ΔPAB and ΔPAC. In these two triangles,



$PB = PC$ (Given)

$\angle PBA = \angle PCA = 90°$ (Given)

$PA = PA$ (Common side)

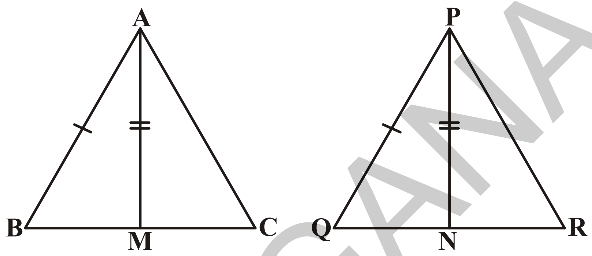
So, $\Delta PAB \cong \Delta PAC$ (RHS rule)

So, $\angle PAB = \angle PAC$ (CPCT)

EXERCISE - 7.3

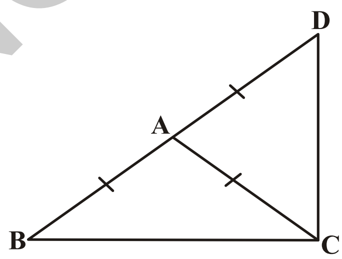
1. AD is an altitude of an isosceles triangle ABC in which AB = AC.
   Show that, (i) AD bisects BC (ii) AD bisects $\angle$A.

2. Two sides AB, BC and median AM of one triangle ABC are respectively equal to sides PQ and QR and median PN of ΔPQR (See figure). Show that:
   (i) $\Delta ABM \cong \Delta PQN$
   (ii) $\Delta ABC \cong \Delta PQR$



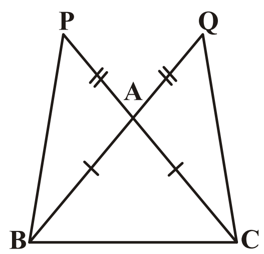
3. BE and CF are two equal altitudes of a triangle ABC. Using RHS congruence rule, prove that the triangle ABC is isosceles.

4. ΔABC is an isosceles triangle in which AB = AC. Show that $\angle B = \angle C$.
   (Hint : Draw AP $\perp$ BC) (Use RHS congruence rule)



5. ΔABC is an isosceles triangle in which AB = AC. Side BA is produced to D such that AD = AB (see figure). Show that $\angle$BCD is a right angle.

6. ABC is a right angled triangle in which $\angle A = 90^0$ and AB = AC. Show that $\angle B = \angle C$.

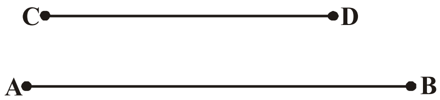


7. Show that the angles of an equilateral triangle are 60^0 each.

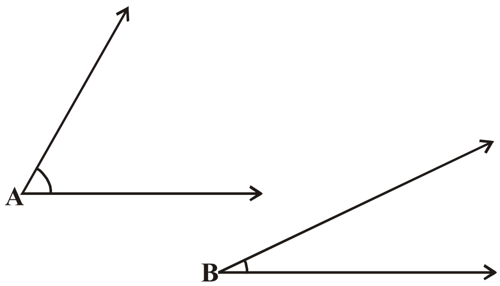
8. In the adjacent figure ΔABC is isosceles as AB=AC, BA and CA and CA are produced to Q and P such that AQ=AP. Show that PB=QC.
   (**Hint :** Compare ΔAPB and ΔAQC)

7.6 INEQUALITIES IN A TRIANGLE

So far, you have been studying the equality of sides and angles of a triangle or triangles. Sometimes, we do come across unequal figures and we need to compare them. For example, line segment AB is greater in length as compared to line segment CD in figure (i) and $\angle$A is greater than $\angle$B in following figure (ii).



(i)



(ii)

Let us now examine whether there is any relation between unequal sides and unequal angles of a triangle. For this, let us perform the following activity:

Activity

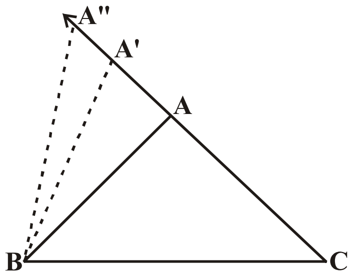
1. Draw a triangle ABC mark a point A' on CA produced (new position of it)

So, $A'C > AC$ (Comparing the lengths)

Join A' to B and complete the triangle $A'BC$.

What can you say about $\angle A'BC$ and $\angle ABC$?

Compare them. What do you observe?



Clearly, $\angle A'BC > \angle ABC$

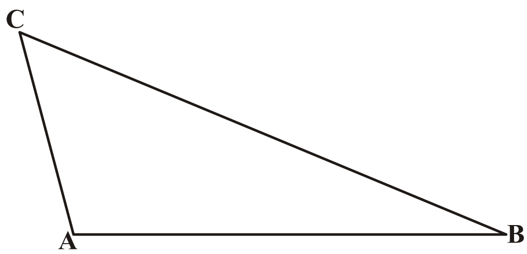
Continue to mark more points on CA (extended) and draw the triangles with the side BC and the points marked.

You will observe that as the length of the side AC is increases (by taking different positions of A), the angle opposite to it, that is, $\angle B$ also increases.

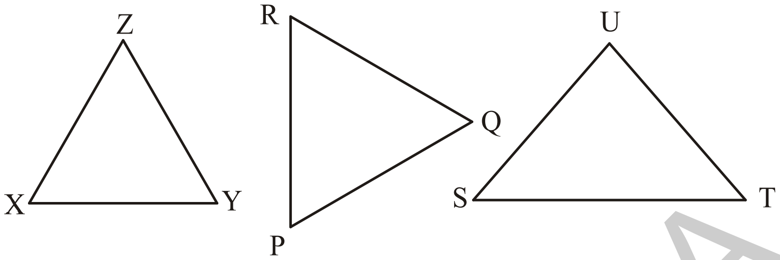
Let us now perform another activity-

2. Construct a scalene triangle ABC (that is a triangle in which all sides are of different lengths). Measure the lengths of the sides.

Now, measure the angles. What do you observe?

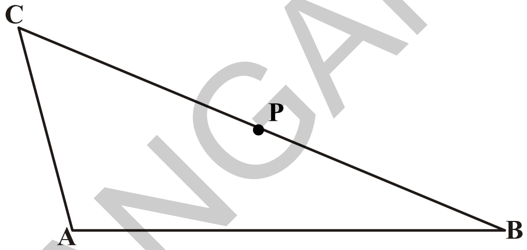


In ΔABC Figure, BC is the longest side and AC is the shortest side. Also, $\angle A$ is the largest and $\angle B$ is the smallest.



Measure angles and sides of each of the above triangles, what is the relation between a side and its opposite angle when campared with another pair?

Theorem-7.6 : If two sides of a triangle are unequal, the angle opposite to the longer side is larger (or greater).

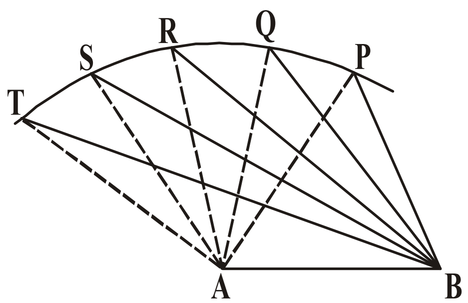


You may prove this theorem by taking a point P on BC such that CA = CP as shown in adjacent figure.

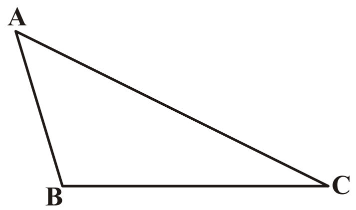
Now, let us do another activity:

Activity

Draw a line-segment AB. A as centre draw an arc with some radius. Mark different points P, Q, R, S and T on it.



Join each of these points with A as well as with B (see figure). Observe that as we move from P to T, $\angle A$ is becoming larger and larger. What is happening to the length of the side opposite to it? Observe that the length of the side is also increasing; that is $\angle TAB > \angle SAB > \angle RAB > \angle QAB > \angle PAB$ and $TB > SB > RB > QB > PB$.

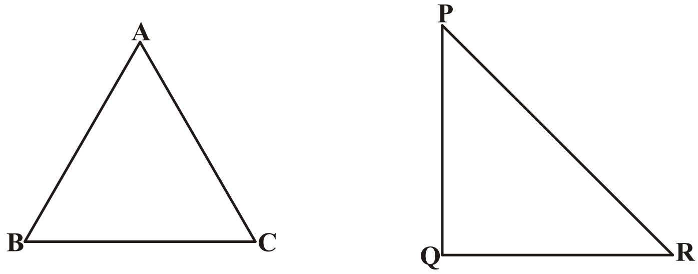


Now, draw any triangle with all angles unequal to each other. Measure the lengths of the sides (see figure).

Observe that the side opposite to the largest angle is the longest. In figure, $\angle B$ is the largest angle and AC is the longest side.

Repeat this activity for some more triangles and we see that the converse of the above Theorem is also true.

Measure angles and sides of each triangle given below. What relation you can visualize for a side and its opposite angle in each triangle.



In this way, we arrive at the following theorem.

Theorem -7.7 : In any triangle, the side opposite to the larger (greater) angle is longer.

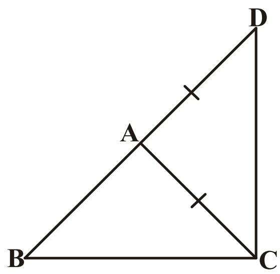
This theorem can be proved by the method of contradiction.

Do This

Now draw a triangle ABC and measure its sides. Find the sum of the sides AB + BC, BC + AC and AC + AB, compare it with the length of the third side. What do you observe?

You will observe that AB + BC > AC, BC + AC > AB and AC + AB > BC. Repeat this activity with other triangles and with this you can arrive at the following theorem:

Theorem-9.8 : The sum of any two sides of a triangle is greater than the third side.

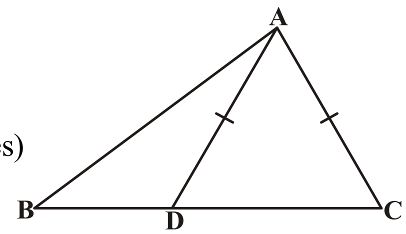


In adjacent figure, observe that the side BA of ΔABC has been produced to a point D such that AD = AC. Can you show that $\angle BCD > \angle BDC$ and BA + AC > BC?

Have you arrived at the proof of the above theorem.

Let us take some examples based on these results.

Example-15. In ΔABC, D is a point on side BC ΔABC such that AD = AC (see figure). Show that AB > AD.



Solution : In ΔDAC,

AD = AC (Given)

So, $\angle ADC = \angle ACD$ (Angles opposite to equal sides)

Now, $\angle ADC$ is an exterior angle for ΔABD.

So, $\angle ADC > \angle ABD$

or, $\angle ACD > \angle ABD$

or, $\angle ACB > \angle ABC$

So, $AB > AC$ (Side opposite to larger angle in ΔABC)

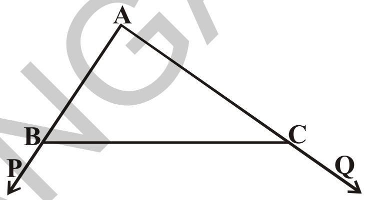
or, $AB > AD$ $(AD = AC)$

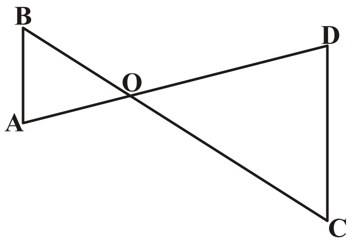
Exercise - 7.4

1. Show that in a right angled triangle, the hypotenuse is the longest side.

2. In adjacent figure, sides AB and AC of ΔABC are extended to points P and Q respectively.

   Also, $\angle PBC < \angle QCB$. Show that $AC > AB$.

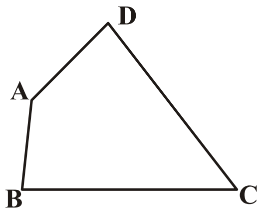


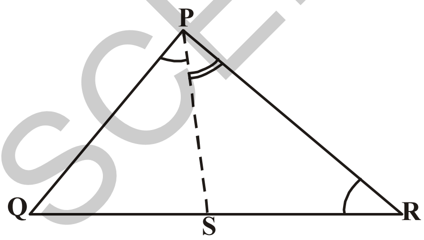


3. In adjacent figure, $\angle B < \angle A$ and $\angle C < \angle D$. Show that $AD < BC$.

4. AB and CD are respectively the smallest and longest sides of a quadrilateral ABCD (see adjacent figure).

   Show that $\angle A > \angle C$ and $\angle B > \angle D$.





5. In adjacent figure, $PR > PQ$ and PS is a angle bisector of $\angle QPR$. Prove that $\angle PSR > \angle PSQ$.

6. If two sides of a triangle measure 4cm and 6cm find all possible measurements (positive Integers) of the third side. How many distinct triangles can be obtained?

7. Try to construct a triangle with 5cm, 8cm and 1cm. Is it possible or not? Why? Give your justification.

WHAT HAVE WE DISCUSSED?

- Figures which are identical i.e. having same shape and size are called congruent figures.
- Three independent measurements are required to make a unique triangle.
- Two triangles are congruent if corresponding angles are congruent and corresponding sides are equal.
- Also, there is a one-one correspondence between the vertices.
- In Congruent triangles corresponding parts are equal and we write in short 'CPCT' for corresponding parts of congruent triangles.
- SAS congruence rule: Two triangles are congruent if two sides and the included angle of one triangle are equal to the corresponding two sides and the included angle of the other triangle.
- ASA congruence rule: Two triangles are congruent if two angles and the included side of one triangle are equal to two angles and the included side of other triangle.
- Angles opposite to equal sides of an isosceles triangle are equal.
- Conversely, sides opposite to equal angles of a triangle are equal.
- SSS congruence rule: If three sides of one triangle are equal to the three sides of another triangle, then the two triangles are congruent.
- RHS congruence rule: If in two right triangles the hypotenuse and one side of one triangle are equal to the hypotenuse and one side of the other triangle, then the two triangles are congruent.
- If two sides of a triangle are unequal, the angle opposite to the longer side is larger.
- In any triangle, the side opposite to the larger angle is longer.
- The sum of any two sides of a triangle is greater than the third side.
- Representation of Line, linesegment and ray

  LM = Length of Linesegment LM; $\overline{LM}$ = Line segment LM $\overrightarrow{LM}$ = Ray LM; $\overleftrightarrow{LM}$ = Line LM.

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Chapter

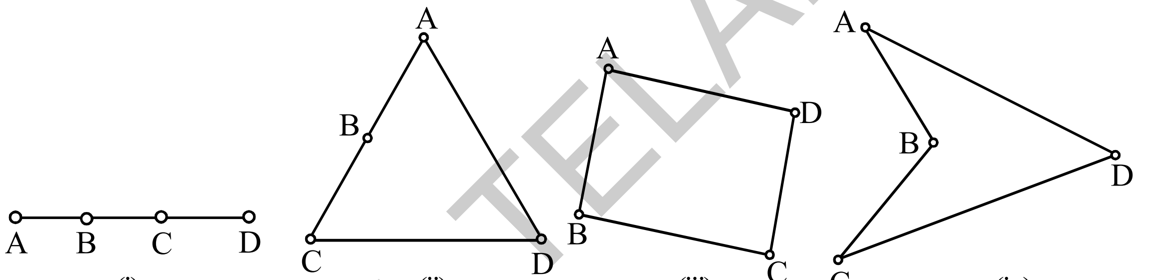
8

Quadrilaterals

N3N2N8

8.1 Introduction

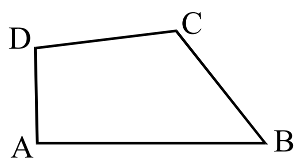
You have learnt some properties of triangles in the previous chapter with justification. You know that a triangle is a figure obtained by joining three non-collinear points in pairs. Do you know which figure you obtain with four points in a plane ? Note that if all the points are collinear, we obtain a line segment (Fig. (i)), if three out of four points are collinear, we get a triangle (Fig(ii)) and if any three points are not collinear, we obtain a closed figure with four sides (Fig (iii), (iv)), we call such a figure as a quadrilateral.



(i) (ii) (iii) (iv)

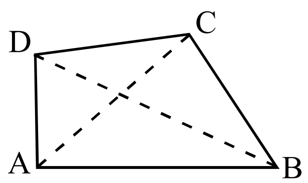
You can easily draw many more quadrilaterals and identify many around you. The Quadrilateral formed in Fig (iii) and (iv) are different in one important aspect. How are they different?

In this chapter we will study quadrilaterals only of type (Fig (iii)). These are convex quadrilaterals.



A quadrilateral is a simple closed figure bounded by four line segments in a plane.

The quadrilateral ABCD has four sides AB, BC, CD and DA, four vertices are A, B, C and D. $\angle A$, $\angle B$, $\angle C$ and $\angle D$ are the four angles formed at the vertices.

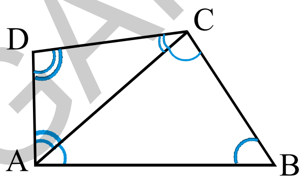


When we join the opposite vertices A, C and B, D, then AC and BD are the two diagonals of the Quadrilateral ABCD.

8.2 PROPERTIES OF A QUADRILATERAL

There are four angles in the interior of a quadrilateral. Can we find the sum of these four angles? Let us recall the angle sum property of a triangle. We can use this property in finding sum of four interior angles of a quadrilateral.

ABCD is a quadrilateral and AC is a diagonal (see figure).



We know the sum of the three angles of ΔABC is,

$\angle BAC + \angle B + \angle ACB = 180^o$...(1) (Angle sum property of a triangle)

Similarly, in ΔADC,

$$\angle CAD + \angle D + \angle DCA = 180^o \quad ...(2)$$

Adding (1) and (2), we get

$$\angle BAC + \angle B + \angle BCA + \angle CAD + \angle D + \angle DCA = 180^0 + 180^0$$

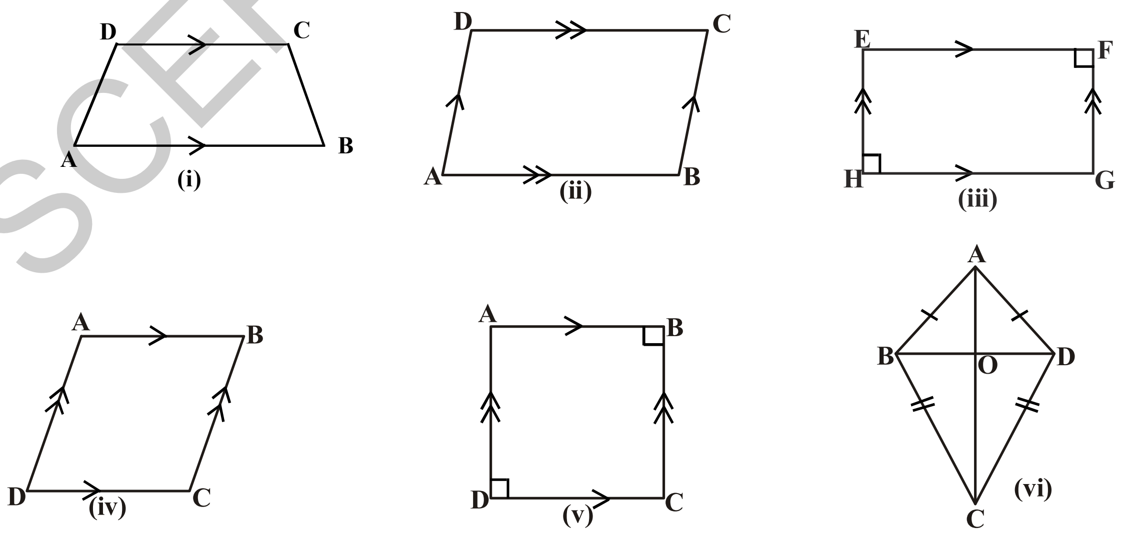
But $\angle BAC + \angle CAD = \angle A$ and $\angle BCA + \angle DCA = \angle C$

So, $\angle A + \angle B + \angle C + \angle D = 360^o$

i.e the sum of four angles of a quadrilateral is 360^o or 4 right angles.

8.3 DIFFERENT TYPES OF QUADRILATERALS

Look at the quadrilaterals drawn below. We have come across most of them earlier. We will quickly consider these and recall their specific names based on their properties.



We observe that

- In fig. (i) the quadrilateral ABCD had one pair of opposite sides AB and DC parallel to each other. Such a quadrilateral is called a **trapezium**.

  If in a trapezium non parallel sides are equal, then the trapezium is an **isosceles trapezium**.

- In fig. (ii) both pairs of opposite sides of the quadrilateral are parallel such a quadrilateral is called a **parallelogram**. Fig.(iii), (iv) and (v) are also parallelograms.
- In fig. (iii) parallelogram EFGH has all its angles as right angles and is called a **rectangle.**
- In fig. (iv) parallelogram has its adjacent sides equal and is called a **rhombus**.
- In fig. (v) parallelogram has its adjacent sides equal and angles of 90° this is called **a square**.
- The quadrilateral ABCD in fig.(vi) has the two pairs of adjacent sides equal, i.e. AB = AD and BC = CD. It is called **a kite**.

Consider what Nisha says:

A rhombus may or maynot be a square but all squares are rhombuses.

Lalita Adds

All rectangles are parallelograms but all parallelograms are not rectangles.

Which of these statements you agree with?

Give reasons for your answer. Write other such statements about different types of quadrilaterals.

Illustrative examples

Example-1 : ABCD is a parallelogram and $\angle A = 60^o$. Find the remaining angles.

Solution : The opposite angles of a parallelogram are equal.

So in a parallelogram ABCD

$\angle C = \angle A = 60^o$ and $\angle B = \angle D$

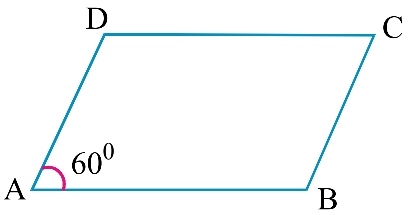
and the sum of adjacent angles of parallelogram is equal to 180^o.

As $\angle A$ and $\angle B$ are adjacent angles

$\angle D = \angle B = 180^o - \angle A$

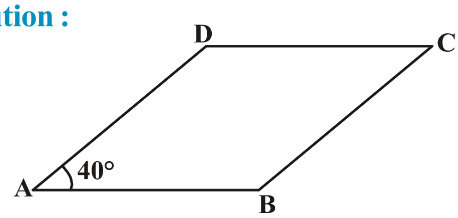
$= 180^o - 60^o = 120^o$.

Thus the remaining angles are 120^o, 60^o, 120^o.



Example-2. In a parallelogram ABCD, $\angle DAB = 40^{o}$ find the remaining angles of the parallelogram.

Solution :



ABCD is a parallelogram

$\angle DAB = \angle BCD = 40°$ and AD || BC

As sum of consecutive angles

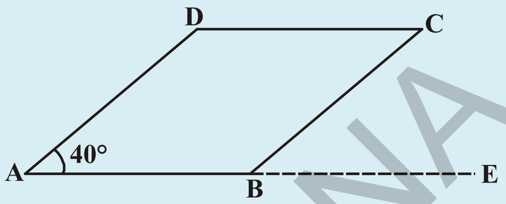
$\angle CBA + \angle DAB = 180°$

$\therefore \quad \angle CBA = 180 - 40°$

$= 140°$

From this we can find $\angle ADC = 140°$ and $\angle BCD = 40°$

Try This



Extend AB to E. Find $\angle CBE$. What do you notice? What kind of angles are $\angle ABC$ and $\angle CBE$?

Example-3 : Two adjacent sides of a parallelogram are 4.5 cm and 3 cm. Find its perimeter.

Solution : Since the opposite sides of a parallelogram are equal the other two sides are 4.5 cm and 3 cm.

Hence, the perimeter = 4.5 + 3 + 4.5 + 3 = 15 cm.

Example-4 : In a parallelogram ABCD, the bisectors of the adjacent $\angle A$ and $\angle B$ intersect at P. Show that $\angle APB = 90^{o}$.

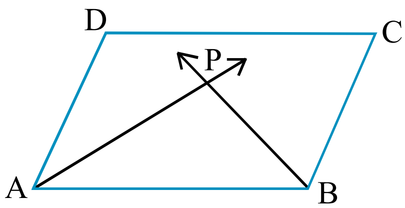
Solution : ABCD is a parallelogram $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are bisectors of adjacent angles, $\angle A$ and $\angle B$.

As, the sum of adjacent angles of a parallelogram is supplementary.

$\angle A + \angle B = 180^{o}$

$\frac{1}{2}\angle A + \frac{1}{2}\angle B = \frac{180}{2}$

$\Rightarrow \angle PAB + \angle PBA = 90°$



In ΔAPB,

$\angle PAB + \angle APB + \angle PBA = 180°$ (angle sum property of triangle)

$\angle APB = 180^{0} - (\angle PAB + \angle PBA)$

$= 180^{o} - 90^{o}$

$= 90^{o}$

Hence proved.

Exercise - 8.1

1. State whether the statements are True or False.
   - (i) Every parallelogram is a trapezium ( )
   - (ii) All parallelograms are quadrilaterals ( )
   - (iii) All trapeziums are parallelograms ( )
   - (iv) A square is a rhombus ( )
   - (v) Every rhombus is a square ( )
   - (vi) All parallelograms are rectangles ( )
2. Complete the following table by writing (YES) if the property holds for the particular Quadrilateral and (NO) if property does not holds.

Properties	Trapezium	Parallelogram	Rhombus	Rectangle	Square
a. Only one pair of opposite sides are parallel	YES				
b. Two pairs of opposite sides are parallel					
c. Opposite sides are equal					
d. Opposite angles are equal					
e. Adjacent angles are supplementary					
f. Diagonals bisect each other					
g. Diagonals are equal					
h. All sides are equal					
i. Each angle is a right angle					
j. Diagonals are perpendicular to each other.					

3. ABCD is trapezium in which AB || CD. If AD = BC, show that $\angle A = \angle B$ and $\angle C = \angle D$.
4. The four angles of a quadrilateral are in the ratio 1: 2:3:4. Find the measure of each angle of the quadrilateral.
5. ABCD is a rectangle AC is diagonal. Find the nature of ΔACD.

8.4 Parallelogram and their Properties

We have seen parallelograms are quadrilaterals. In the following we would consider the properties of parallelograms.

Activity

Cut-out a parallelogram from a sheet of paper again and cut along one of its diagonal. What kind of shapes you obtain? What can you say about these triangles?

Place one triangle over the other. Can you place each side over the other exactly. You may need to turn the triangle around to match sides. Since, the two traingles match exactly they are congruent to each other.

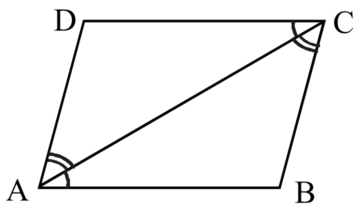
Do this with some more parallelograms. You can select any diagonal to cut along.

We see that each diagonal divides the parallelogram into two congruent triangles.

Let us now prove this result.

Theorem- 8.1 : A diagonal of a parallelogram divides it into two congruent triangles.

Proof: Consider the parallelogram ABCD.



Join A and C. AC is a diagonal of the parallelogram.

Since AB || DC and AC is transversal

$\angle DCA = \angle CAB$. (Interior alternate angles)

Similarly DA || CB and AC is a transversal therefore $\angle DAC = \angle BCA$.

We have in ΔACD and ΔCAB

$\angle DCA = \angle CAB$ and $\angle DAC = \angle BCA$

also AC = CA. (Common side)

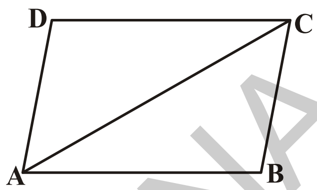
Therefore $\Delta ABC \cong \Delta CDA$.

This means that the two traingles are congruent by ASA rule (angle, side and angle). This means that diagonal AC divides the parallelogram in two congruent traingles.

Theorem-8.2 : In a parallelogram, opposite sides are equal and opposite angles are equal.

Proof: We have already proved that a diagonal of a parallelogram divides it into two congruent triangles.

Thus in figure $\Delta ACD \cong \Delta CAB$



We have therefore $AB = DC$ and $\angle CBA = \angle ADC$

also $AD = BC$ and $\angle DAC = \angle ACB$

$\angle CAB = \angle DCA$

$\therefore \angle ACB + \angle DCA = \angle DAC + \angle CAB$

i.e. $\angle DCB = \angle DAB$

From above in a parallelogram

i. The opposite sides are equal.

ii. The opposite angles are equal.

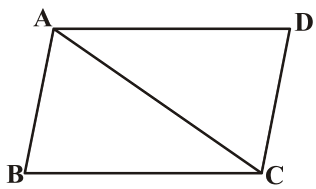
We have proved that in a convex quadrilateral if opposite sides are parallel then the opposite sides are equal and opposite angles are equal.

We will now try to prove its converse i.e. if the opposite sides of a quadrilateral are equal, then it is a parallelogram.

Theorem-8.3 : If each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram.

Proof : Consider the quadrilateral ABCD with $AB = DC$ and $BC = AD$.

Draw a diagonal AC.



Consider ΔABC and ΔCDA

We have $BC = AD$, $AB = DC$ and $AC = CA$ (Common side)

So $\Delta ABC \cong \Delta CDA$ (why?)

Therefore $\angle BCA = \angle DAC$ with AC as transversal

$\therefore AB \parallel DC$ ...(1)

Since $\angle ACD = \angle CAB$ with CA as transversal

We have $BC \parallel AD$ ...(2)

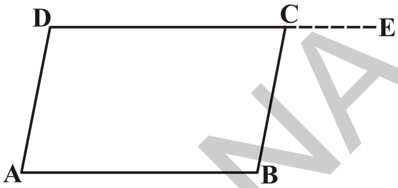
Therefore, ABCD is a parallelogram. By (1) and (2)

You have just seen that in a parallelogram both pairs of opposite sides are equal and conversely if both pairs of opposite sides of a quadrilateral are equal, then it is a parallelogram.

Can we show the same for a quadrilateral for which the pairs of opposite angles are equal?

Theorem-8.4 : In a quadrilateral, if each pair of opposite angles are equal then it is a parallelogram.

Proof: In a quadrilateral ABCD, $\angle A = \angle C$ and $\angle B = \angle D$ then we need to prove that ABCD is a parallelogram.



We know $\angle A + \angle B + \angle C + \angle D = 360°$

$$\angle A + \angle B = \angle C + \angle D = \frac{360°}{2}$$

i.e. $\angle A + \angle B = 180°$

Extend DC to E

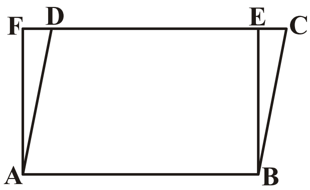
$\angle BCD + \angle BCE = 180°$ hence $\angle BCE = \angle ADC$

If $\angle BCE = \angle ADC$ then AD || BC (Why?)

With DC as a transversal

We can similarly show AB || DC or ABCD is a parallelogram.

Exercise - 8.2



1. In the adjacent figure ABCD is a parallelogram and ABEF is a rectangle show that $\Delta AFD \cong \Delta BEC$.
2. Show that the diagonals of a rhombus divide it into four congruent triangles.
3. In a quadrilateral ABCD, the bisector of $\angle C$ and $\angle D$ intersect at O.

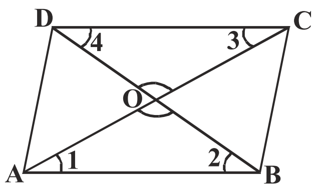
   Prove that $\angle COD = \frac{1}{2}(\angle A + \angle B)$

8.5 Diagonals of a Parallelogram

Theorem-8.5 : The diagonals of a parallelogram bisect each other.

Proof: Draw a parallelogram ABCD.

Draw both of its diagonals AC and BD their intersecting point is at 'O'.



In ΔOAB and ΔOCD

Mark the angles formed as $\angle 1, \angle 2, \angle 3, \angle 4$

$\angle 1 = \angle 3$ (AB || CD and AC transversal)

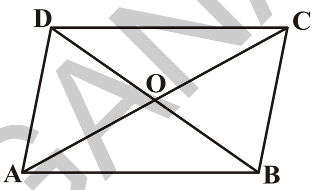
$\angle 2 = \angle 4$ (Why) (Interior alternate angles) and $AB = CD$ (opposite sides of parallelogram)

By ASA rule

$\Delta OCD \cong \Delta OAB$

$CO = OA$, $DO = OB$ or diagonals bisect each other. (CPCT)

Hence we have to check if the converse is also true. Converse is if diagonals of a quadrilateral bisect each other then it is a parallelogram.



Theorem-8.6 : If the diagonals of a quadrilateral bisect each other then it is a parallelogram.

Proof: ABCD is a quadrilateral.

AC and BD are the diagonals intersecting at 'O',

such that $OA = OC$ and $OB = OD$.

Prove that ABCD is a parallelogram.

(**Note** : Consider ΔAOB and ΔCOD. Are these triangles congruent? If so how ABCD become a parallelogram?)

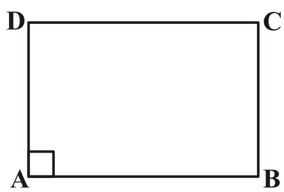
8.5.1 More geometrical statements

In the previous examples we have shown that starting from general statement we can make many statements about a particular figure(Parallelogram). We use previous results to deduce new statements. Note that these statements need not be verified by measurements as they have been proved logically.

Such statements that are deduced from the previously known and proved statements are called corollary. A corollary is a statement, the truth of which follows readily from an established theorem.

Corollary-1 : Show that each angle of a rectangle is a right angle.

Solution : Rectangle is a parallelogram in which one angle is a right angle.



ABCD is a rectangle. Let one angle is $\angle A = 90^o$

We have to show that $\angle B = \angle C = \angle D = 90^o$

Proof : Since ABCD is a parallelogram,

thus $AD \parallel BC$ and AB is a transversal

so $\angle A + \angle B = 180^o$ (Interior angles on the same side of a transversal)

as $\angle A = 90^o$ (given)

$\therefore \angle B = 180^o - \angle A$

$= 180^o - 90^o = 90^o$

Now $\angle C = \angle A$ and $\angle D = \angle B$ (opposite angles of parallelogram)

So $\angle C = 90^o$ and $\angle D = 90^o$.

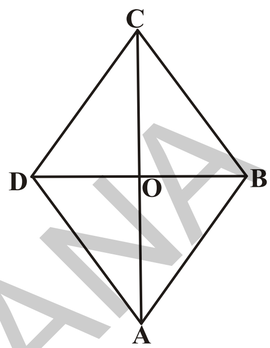
Therefore each angle of a rectangle is a right angle.

Corollary-2 : Show that the diagonals of a rhombus are perpendicular to each other.

Proof : A rhombus is a parallelogram with all sides equal.

ABCD is a rhombus, diagonals AC and BD intersect at O

We want to show that AC is perpendicular to BD



Consider ΔAOB and ΔBOC

OA = OC (Diagonals of a parallelogram bisect each other)

OB = OB (common side to ΔAOB and ΔBOC)

AB = BC (sides of rhombus)

Therefore $\Delta AOB \cong \Delta BOC$ (S.S.S rule)

So $\angle AOB = \angle BOC$

But $\angle AOB + \angle BOC = 180^\circ$ (Linear pair)

Therefore $2\angle AOB = 180^\circ$

or $\angle AOB = \dfrac{180^\circ}{2} = 90^\circ$

Similarly $\angle BOC = \angle COD = \angle AOD = 90^\circ$ (Same angle)

Hence AC is perpendicular on BD

So, the diagonals of a rhombus are perpendicular to each other.

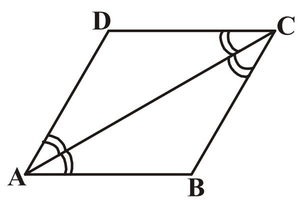
Corollary-3 : In a parallelogram ABCD, if the diagonal AC bisects the angle $\angle A$, then ABCD is a rhombus.

Proof : ABCD is a parallelogram

Therefore AB || DC. AC is the transversal intersects $\angle A$ and $\angle C$

So, $\angle BAC = \angle DCA$ (Interior alternate angles) ...(1)

$\angle BCA = \angle DAC$ ...(2)



But it is given that AC bisects $\angle A$

So $\angle BAC = \angle DAC$

$\therefore \angle DCA = \angle DAC$ ...(3)

Thus AC bisects $\angle C$ also

From (1), (2) and (3), we have

$\angle BAC = \angle BCA$

In ΔABC, $\angle BAC = \angle BCA$ means that $BC = AB$ (isosceles triangle)

But $AB = DC$ and $BC = AD$ (opposite sides of the parallelogram ABCD)

$\therefore$ $AB = BC = CD = DA$

Hence, ABCD is a rhombus.

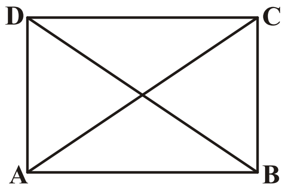
Corollary-4 : Show that the diagonals of a rectangle are of equal length.

Proof : ABCD is a rectangle and AC and BD are its diagonals

We want to show $AC = BD$

ABCD is a rectangle, means ABCD is a parallelogram with all its angles equal to right angle.

Consider the triangles ΔABC and ΔBAD,



$AB = BA$ (Common)

$\angle B = \angle A = 90^o$ (Each angle of rectangle)

$BC = AD$ (opposite sides of the rectangle)

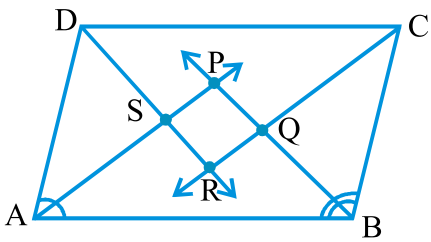
Therefore, $\Delta ABC \cong \Delta BAD$ (S.A.S rule)

This implies that $AC = BD$

or the diagonals of a rectangle are equal.

Corollary-5 : Show that the angle bisectors of a parallelogram form a rectangle.

Proof : ABCD is a parallelogram. The bisectors of angles $\angle A$, $\angle B$, $\angle C$ and $\angle D$ intersect at P, Q, R, S to form a quadrilateral. (See adjacent figure)



Since ABCD is a parallelogram, $AD \parallel BC$. Consider AB as transversal intersecting them then $\angle BAD + \angle ABC = 180°$ (Consecutive angles of Parallelogram)

We know $\angle BAP = \frac{1}{2} \angle BAD$ and $\angle ABP = \frac{1}{2} \angle ABC$ [Since $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are the bisectors of $\angle A$ and $\angle B$ respectively]

$$\Rightarrow \frac{1}{2}\angle BAD + \frac{1}{2}\angle ABC = \frac{1}{2} \times 180^o$$

Or $\angle BAP + \angle ABP = 90^o$ ...(1)

But In ΔAPB,

$$\angle BAP + \angle APB + \angle ABP = 180^\circ \text{ (Angle sum property of the triangle)}$$

So $\angle APB = 180^\circ - (\angle BAP + \angle ABP)$

$\Rightarrow \angle APB = 180^\circ - 90^\circ$ (From (1))

$= 90^\circ$

We can see that $\angle SPQ = \angle APB = 90^\circ$

Similarly, we can show that $\angle CRD = \angle QRS = 90^\circ$ (Same angle)

But $\angle BQC = \angle PQR$ and $\angle DSA = \angle PSR$ (Why?)

$\therefore$ $\angle PQR = \angle QRS = \angle PSR = \angle SPQ = 90^\circ$

Hence PQRS has all the four angles equal to 90°.

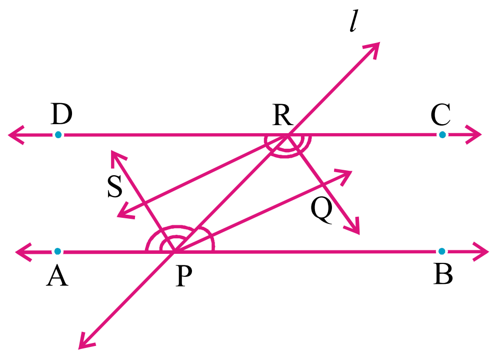
We can therefore say PQRS is a rectangle.

THINK, DISCUSS AND WRITE

1. Show that the diagonals of a square are equal and **right bisectors** of each other.
2. Show that the diagonals of a rhombus divide it into four congruent triangles.

Some Illustrative examples

Example-5. $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ are two parallel lines and a transversal l, intersects $\overleftrightarrow{AB}$ at P and $\overleftrightarrow{DC}$ at R. Prove that the bisectors of the interior angles form a rectangle.



Proof : $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, l is the transversal intersecting $\overleftrightarrow{AB}$ at P and $\overleftrightarrow{DC}$ at R respectively.

Let $\overrightarrow{PQ}$, $\overrightarrow{RQ}$, $\overrightarrow{RS}$ and $\overrightarrow{PS}$ are the bisectors of $\angle RPB$, $\angle CRP$, $\angle DRP$ and $\angle APR$ respectively.

$\angle BPR = \angle DRP$ (Interior Alternate angles) ...(1)

$$\left.\begin{aligned} &\text{But } \angle RPQ = \frac{1}{2}\angle BPR \quad (\because \overrightarrow{PQ} \text{ is the bisector of } \angle BPR) \\ &\text{also } \angle PRS = \frac{1}{2}\angle DRP \quad (\because \overrightarrow{RS} \text{ is the bisector of } \angle DPR). \end{aligned}\right\} \quad ...(2)$$

From (1) and (2)

$\angle RPQ = \angle PRS$

These are interior alternate angles made by $\overline{PR}$ with the lines $\overrightarrow{PQ}$ and $\overrightarrow{RS}$

$\therefore \quad \overrightarrow{PQ} \parallel \overrightarrow{RS}$

Similarly

$\angle PRQ = \angle RPS$, hence $\overrightarrow{PS} \parallel \overrightarrow{RQ}$

Therefore PQRS is a parallelogram ... (3)

We have $\angle BPR + \angle CRP = 180^o$ (interior angles on the same side of the transversal l with line $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$)

$$\frac{1}{2}\angle BPR + \frac{1}{2}\angle CRP = 90^o$$

$$\Rightarrow \angle RPQ + \angle PRQ = 90^o$$

But in Δ PQR,

$\angle RPQ + \angle PQR + \angle PRQ = 180^o$ (three angles of a triangle)

$\angle PQR = 180^o - (\angle RPQ + \angle PRQ)$

$= 180^o - 90^o = 90^o$ ... (4)

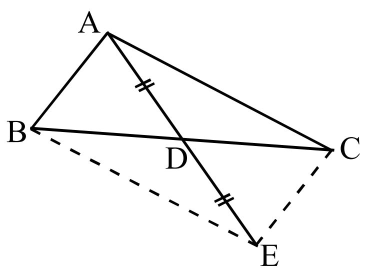
From (3) and (4)

PQRS is a parallelogram with one of its angles as a right angle.

Hence PQRS is a rectangle

Example-6. In a triangle ABC, AD is the median drawn on the side BC is produced to E such that AD = ED prove that ABEC is a parallelogram.

Proof : AD is the median of Δ ABC



Produce AD to E such that AD = ED

Join BE and CE.

Now In Δ^s ABD and ECD

BD = DC (D is the midpoints of BC)

$\angle ADB = \angle EDC$ (vertically opposite angles)

AD = ED (Given)

So $\Delta ABD \cong \Delta ECD$ (SAS rule)

Therefore, AB = CE (CPCT)

also $\angle ABD = \angle ECD$

These are interior alternate angles made by the transversal $\overleftrightarrow{BC}$ with lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CE}$.

$\therefore \overleftrightarrow{AB} \parallel \overleftrightarrow{CE}$

Thus, in a Quadrilateral ABEC,

AB || CE and AB = CE

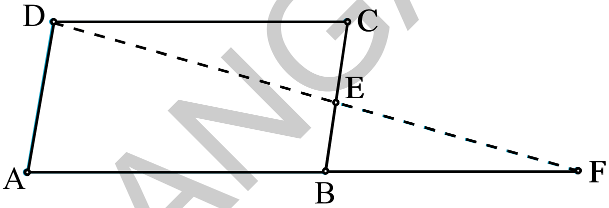
Hence ABEC is a parallelogram.

Exercise - 8.3

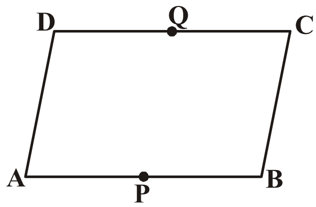
1. The opposite angles of a parallelogram are $(3x - 2)^{o}$ and $(x + 48)^{o}$.
   Find the measure of each angle of the parallelogram.

2. Find the measure of all the angles of a parallelogram, if one angle is 24^{o} less than the twice of the smallest angle.

3. In the adjacent figure ABCD is a parallelogram and E is the midpoint of the side BC. If AB and DE are produced to meet at F, show that AF = 2AB.



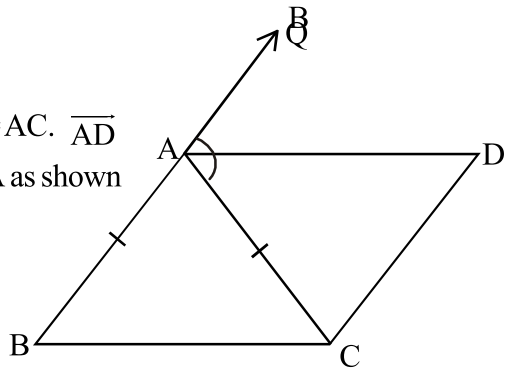
4. In the adjacent figure ABCD is a parallelogram P and Q are the midpoints of sides AB and DC respectively. Show that PBCQ is also a parallelogram.



5. ABC is an isosceles triangle in which AB = AC. $\overrightarrow{AD}$ bisects exterior angle $\angle QAC$ and CD || BA as shown in the figure. Show that

   (i) $\angle DAC = \angle BCA$

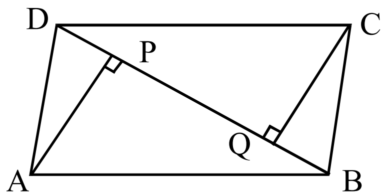
   (ii) ABCD is a parallelogram



6. ABCD is a parallelogram AP and CQ are perpendiculars drawn from vertices A and C on diagonal BD (see figure) show that

   (i) $\Delta APB \cong \Delta CQD$

   (ii) AP = CQ



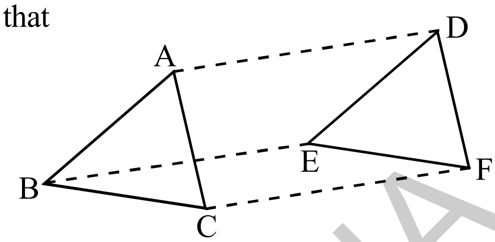
7. In Δ^s ABC and DEF, AB || DE; BC = EF and BC || EF. Vertices A, B and C are joined to vertices D, E and F respectively (see figure). Show that

   (i) ABED is a parallelogram

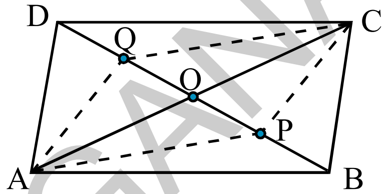
   (ii) BCFE is a parallelogram

   (iii) AC = DF

   (iv) $\Delta ABC \cong \Delta DEF$



8. ABCD is a parallelogram. AC and BD are the diagonals intersect at O. P and Q are the points of tri section of the diagonal BD. Prove that CQ || AP and also AC bisects PQ (see figure).



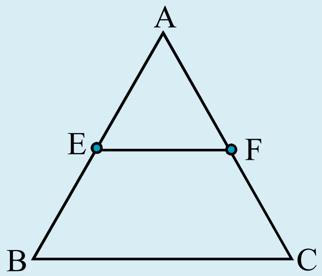
9. ABCD is a square. E, F, G and H are the mid points of AB, BC, CD and DA respectively. Such that AE = BF = CG = DH. Prove that EFGH is a square.

8.6 The Midpoint Theorem of Triangle

We have studied properties of triangle and of a quadrilateral. Let us try and consider the midpoints of the sides of a triangle and what can be derived from them.

Try This

Draw a triangle ABC and mark the midpoints E and F of two sides of triangle. $\overline{AB}$ and $\overline{AC}$ respectively. Join the point E and F as shown in the figure.



Measure EF and the third side BC of the triangle. Also measure $\angle AEF$ and $\angle ABC$.

We find $\angle AEF = \angle ABC$ and $EF = \frac{1}{2} BC$

As these are corresponding angles made by the transversal AB with lines EF and BC, we say EF || BC.

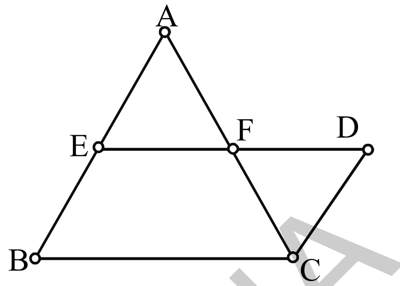
Repeat this activity with some more triangles.

So, we arrive at the following theorem.

Theorem-8.7 : The line segment joining the midpoints of two sides of a triangle is parallel to the third side and also half of it.

Given : ABC is a triangle with E and F as the midpoints of AB and AC respectively.

R.T.P : (i) EF || BC   (ii) $EF = \frac{1}{2}BC$



Proof:- Join EF and extend it, and draw a line parallel to BA through C to meet to produced EF at D.

In Δ^s AEF and ΔCDF

AF = CF (F is the midpoint of AC)

$\angle AFE = \angle CFD$ (vertically opposite angles.)

and $\angle AEF = \angle CDF$ (Interior alternate angles as CD || BA with transversal ED.)

By A.S.A congruency rule

$\therefore$ Δ AEF $\cong$ ΔCDF (ASA congruency rule)

Thus AE = CD and EF = DF (CPCT)

We know AE = BE

Therefore BE = CD

Since BE || CD and BE = CD, BCDE is a parallelogram.

So ED || BC

$\Rightarrow$ EF || BC

As BCDE is a parallelogram, ED = BC(how ?) ($\because$ DF = EF)

we have shown FD = EF

$\therefore$ 2EF = BC

Hence $EF = \frac{1}{2}BC$

We can see that the converse of the above statement is also true. Let us state it and then see how we can prove it.

Theorem-8.8 : The line drawn through the midpoint of one of the sides of a triangle and parallel to another side will bisect the third side

Proof: Draw ΔABC. Mark E as the mid point of side AB. Draw a line *l* passing through E and parallel to BC. The line intersects AC at F.

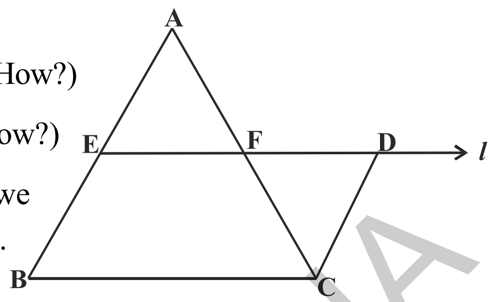
Construct CD || BA

We have to show AF = CF

Consider ΔAEF and ΔCFD

$\angle EAF = \angle DCF$ (BA || CD and AC is transversal) (How?)

$\angle AEF = \angle D$ (BA || CD and ED is transversal) (How?)



We can not prove the congruence of the triangles as we have not shown any pair of sides in the two triangles as equal.

To do so we consider EB || DC

and ED || BC

Thus EDCB is a parallelogram and we have BE = DC.

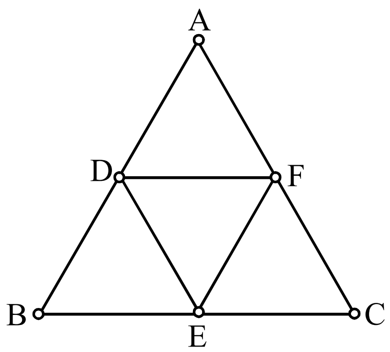
Since BE = AE we have AE = DC.

Hence $\Delta AEF \cong \Delta CFD$ (ASA)

$\therefore$ AF = CF

Some more examples

Example-7. In ΔABC, D, E and F are the midpoints of sides AB, BC and CA respectively. Show that ΔABC is divided into four congruent triangles, when the three midpoints are joined to each other. (ΔDEF is called medial triangle)



Proof : D, E are midpoints of $\overline{AB}$ and $\overrightarrow{AY}$ of triangle ABC respectively

So by Mid-point Theorem,

DE || AC

Similarly DF || BC and EF || AB.

Therefore ADEF, BEFD, CFDE are all parallelograms

In the parallelogram ADEF, DF is the diagonal

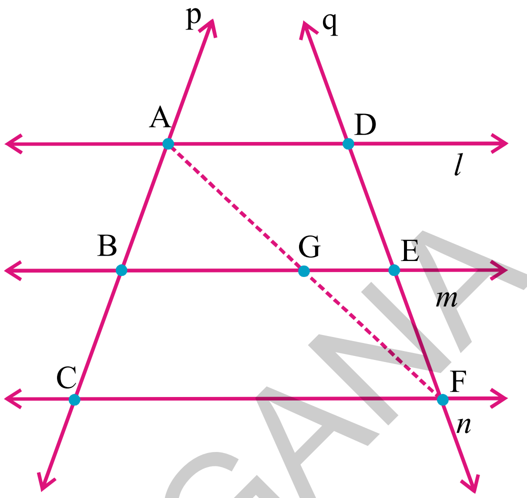
So $\Delta ADF \cong \Delta DEF$ (Diagonal divides the parallelogram into two congruent triangles)

Similarly $\Delta BDE \cong \Delta DEF$

and $\Delta CEF \cong \Delta DEF$

So, all the four triangles are congruent.

We have shown that a triangle ABC is divided in to four congruent triangles by joining the midpoints of the sides.



Example-8. l, m and n are three parallel lines intersected by the transversals p and q at A, B, C and D,E, F such that they make equal intercepts AB and BC on the transversal p. Show that the intercepts DE and EF on q are also equal.

Proof : We need to connect the equality of AB and BC to comparing DE and EF. We join A to F and call the intersection point with 'm' as G.

In ΔACF, AB = BC (given)

Therefore B is the midpoint of AC.

and BG || CF (how ?)

So G is the midpoint of AF (By the theorem).

Now in Δ AFD, we can apply the same reason as G is the midpoint of AF and GE || AD,

E is the midpoint of DF.

Thus DE = EF.

Hence l, m and n cut off equal intercepts on q also.

Example-9. In the Fig. AD and BE are medians of ΔABC and BE || DF. Prove that

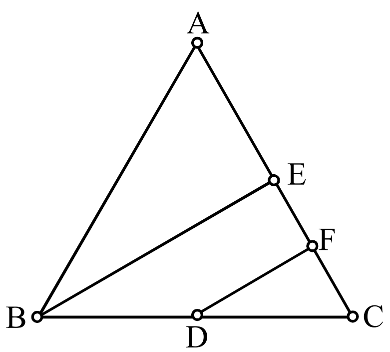
$$CF = \frac{1}{4}AC.$$

Proof : If Δ ABC, D is the midpoint of BC and BE || DF; By Theorem F is the midpoint of CE.

$$\therefore CF = \frac{1}{2}CE$$

$$= \frac{1}{2}\left(\frac{1}{2}AC\right) \quad \text{(How ?)}$$

Hence $CF = \frac{1}{4}AC$.



Example-10. ABC is a triangle and through A, B, C lines are drawn parallel to BC, CA and AB respectively intersecting at P, Q and R. Prove that the perimeter of ΔPQR is double the perimeter of ΔABC.

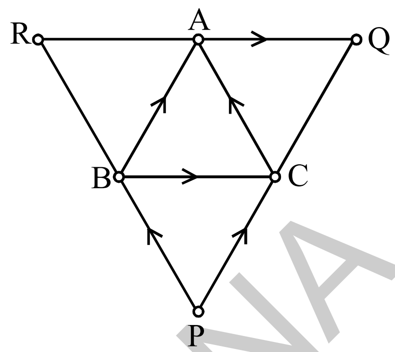
Proof : AB || QP and BC || RQ So ABCQ is a parallelogram.

Similarly BCAR, ABPC are parallelograms

$\therefore$ BC = AQ and BC = RA

$\Rightarrow$ A is the midpoint of QR

Similarly B and C are midpoints of PR and PQ respectively.



$\therefore AB = \frac{1}{2}PQ; \quad BC = \frac{1}{2}QR \quad \text{and } CA = \frac{1}{2}PR$ (How)

(State the related theorem)

Now perimeter of $\Delta PQR = PQ + QR + PR$

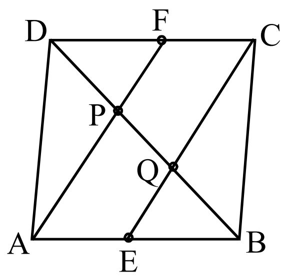
$= 2AB + 2BC + 2CA$

$= 2(AB + BC + CA)$

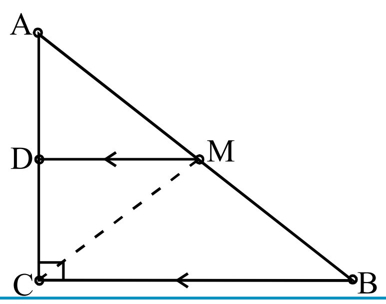
$= 2$ (perimeter of ΔABC).

Exercise - 8.4

1. ABC is a triangle. D is a point on AB such that $AD = \frac{1}{4}AB$ and E is a point on AC such that $AE = \frac{1}{4}AC$. If DE = 2 cm find BC.
2. ABCD is quadrilateral E, F, G and H are the midpoints of AB, BC, CD and DA respectively. Prove that EFGH is a parallelogram.
3. Show that the figure formed by joining the midpoints of sides of a rhombus successively is a rectangle.
4. In a parallelogram ABCD, E and F are the midpoints of the sides AB and DC respectively. Show that the line segments AF and EC trisect the diagonal BD.



5. Show that the line segments joining the midpoints of the opposite sides of a quadrilateral and bisect each other.
6. ABC is a triangle right angled at C. A line through the midpoint M of hypotenuse AB and Parallel to BC intersects AC at D. Show that
   (i) D is the midpoint of AC
   (ii) $MD \perp AC$
   (iii) $CM = MA = \frac{1}{2}AB$.



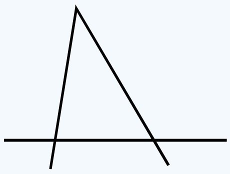
What have we discussed?

1. A quadrilateral is a simple closed figure formed by four line segments in a plane.
2. The sum of four angles in a quadrilateral is 360^0 or 4 right angles.
3. Trapezium, parallelogram, rhombus, rectangle, square and kite are special types of quadrilaterals.
4. Parallelogram is a special type of quadrilateral with many properties. We have proved the following theorems.
    a) The diagonal of a parallelogram divides it into two congruent triangles.
    b) The opposite sides and angles of a parallelogram are equal.
    c) If each pair of opposite sides of a quadrilateral are equal then it is a parallelogram.
    d) If each pair of opposite angles are equal then it is a parallelogram.
    e) Diagonals of a parallelogram bisect each other.
    f) If the diagonals of a quadrilateral bisect each other then it is a parallelogram.
5. Mid point theorm of triangle and converse
    a) The line segment joining the midpoints of two sides of a triangle is parallel to the third side and also half of it.
    b) The line drawn through the midpoint of one of the sides of a triangle and parallel to another side will bisect the third side.

R2T9U7

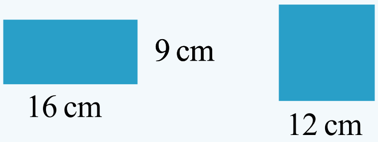
Brain teaser

1. Creating triangles puzzle



Add two straight lines to the above diagram and produce 10 triangles.

2. Take a rectangular sheet of paper whose length is 16 cm and breadth is 9 cm. Cut it in to exactly 2 pieces and join them to make a square.



Chapter

9

Statistics

9.1 Introduction

One day Ashish visited his mathematics teacher at his home. At that time his teacher was busy in compiling the information which he had collected from his ward for the population census of India.

Ashish : Good evening sir, it seems you are very busy. Can I help you in your work, Sir?

Teacher : Ashish, I have collected the household information for census i.e. number of family members, their age group, family income, type of house they live and other data.

Ashish : Sir, what is the use of this information?

Teacher : This information is useful for government in planning the developmental programmes and allocation of resources.

Ashish : How does government use this large information?

Teacher : The Census Department compiles this massive data and by using required data handling tools analyse the data and interpretes the results in the form of information. Ashish, you must have learnt basic statistics (data handling) in your earlier classes, didn't you?

Like Ashish we too come across a lot of situations where we see information in the form of facts, numerical figures, tables, graphs etc. These may relate to price of vegetables, city temperature, cricket scores, polling result and so on. These facts or figures which are numerical or otherwise collected with a definite purpose are called **'data'**. Extraction of meaning from the data is studied in a branch of mathematics called **statistics**.

Lets us first revise what we have studied in statistics (data handling) in our previous classes.

9.2 Collection of Data

The primary activity in statistics is to collect the data with some purpose. To understand this let us begin with an exercise of collecting data by performing the following activity.

ACTIVITY

Divide the students of your class into four groups. Allot each group the work of collecting one of the following kinds of data:

i. Weights of all the students in your class.

ii. Number of siblings that each student have.

iii. Day wise number of absentees in your class during last month.

iv. The distance between the school and home of every student of your class.

Let us discuss how these students have collected the required information?

1. Have they collected the information by enquiring each student directly or by visiting every house personally by the students?

2. Have they got the information from source like data available in school records?

In first case when the information was collected by the investigator (student) with a definite objective, the data obtained directly from the source is called **primary data** (as in (i), (ii), (iv)).

In the above task (iii) number of absentees in the last month could only be known by school attendance register. So here we are using data which is already collected by class teachers. This is called secondary data. The information collected from a source, which had already been recorded, say from registers, is called **secondary data**.

DO THIS

Which of the following are primary and secondary data?

i. Collection of the data about enrollment of students in your school for a period from 2001 to 2010.

ii. Height of students in your class recorded by physical education teacher.

9.3 Presentation of Data

Once the data is collected, the investigator has to find out ways to present it in the form which is meaningful, easy to understand and shows its main features at a glance. Let us take different situations where we need to present the data.

Consider the marks obtained by 15 students in a mathematics test out of 50 marks:

The data in this form is called raw data.

From the given data you can easily identify the minimum and maximum marks. You also remember that the difference between the maximum and minimum of the observations is called the **range** of given data.

Here minimum and maximum marks are 7 and 50 respectively.

So the range = 50 – 7 = 43,

From the above we can also say that our data lies from 7 to 50.

Now let us answer the following questions from the above data.

i. Find the middle value of the given data.

ii. How many children got 60% or more marks in the mathematics test?

Discussion

(i) Ikram said that the middle value of the data is 25 because the exam was conducted for 50 marks. What do you think?

Mary said that it is not the middle value of the data. What do you think?

In this case we have marks of 15 students as raw data, so after arranging the data in ascending order,

7, 11, 20, 20, 25, 28, 30, 34, 39, 40, 42, 42, 45, 50, 50

we can say that the 8^{th} term is the middle term and it is 34.

(ii) You already know how to find 60% of 50 marks (i.e. $\frac{60}{100} \times 50 = 30$).

You find that there are 9 students who got 60% or more marks (i.e. 30 marks or more).

When the number of observations in a data are too many, presentation of the data in ascending or descending order can be quite time consuming. So we have to think of an alternative method.

See the given example.

Example-1. Consider the marks obtained by 50 students in a mathematics test for a total marks of 10.

5, 8, 6, 4, 2, 5, 4, 9, 10, 2, 1, 1, 3, 4, 5,
8, 6, 7, 10, 2, 1, 1, 3, 4,4, 5, 8, 6, 7, 10,
2,8, 6, 4, 2, 5, 4, 9, 10, 2, 1, 1, 3, 4, 5,
8, 6,4, 5, 8

Marks	Tally Marks	No of students
1	𝍸 |	6
2	𝍸 |	6
3	|||	3
4	𝍸 ||||	9
5	𝍸 ||	7
6	𝍸	5
7	||	2
8	𝍸 |	6
9	||	2
10	||||	4
	Total	50

The data is tabulated by using the tally marks, as shown in table.

Recall that the number of students who have obtained a certain number of marks is called the frequency of those marks. For example, 9 students got 4 marks each. So the frequency of 4 marks is 9.

Here in the table, tally marks are useful in tabulating the raw data.

Sum of all frequencies in the table gives the total number of observations of the data.

As the actual observations of the data are shown in the table with their frequencies, this table is called '**Ungrouped Frequency Distribution Table**' or '**Table of Weighted Observations**'.

ACTIVITY

Make frequency distribution table of the initial letters of that denotes surnames of your classmates and answer the following questions.

(i) Which initial letter occured mostly among your classmates?

(ii) How many students names start with the alphabat 'I'?

(iii) Which letter occured least as an initial among your classmates?

Suppose for specific reason, we want to represent the data in three categories (i) how many students need extra classes, (ii) how many have an average performance and (iii) how many did well in the test. Then we can make groups as per the requirement and grouped frequency table as shown below.

Class interval (marks)	Category	Tally marks	No. of students
1 - 3	(Need extra class)	卌 卌 卌	15
4 - 5	(Average)	卌 卌 卌 |	16
6 - 10	(Well)	卌 卌 卌 ||||	19

To classify the data according to the requirement or if there are large number of observations. We make groups to condense it. Let's take one more example in which group and frequency make us easy to understand the data.

Example-2. The weight (in grams) of 50 oranges, picked at random from a basket of oranges, are given below:

35, 45, 55, 50, 30, 110, 95, 40, 70, 100, 60, 80, 85, 60, 52, 95, 98, 35, 47, 45, 105, 90, 30, 50, 75, 95, 85, 80, 35, 45, 40, 50, 60, 65, 55, 45, 30, 90, 115, 65, 60, 40, 100, 55, 75, 110, 85, 95, 55, 50

To present such a large amount of data and to make sense of it, we make groups like 30-39, 40-49, 50-59, ... 100-109, 110-119. (since our data is from 30 to 115). These groups are called 'classes' or class-intervals, and their size is called length of the class or class width, which is 10 in this case. In each of these classes the least number is called the lower limit and the greatest number is called the upper limit, e.g. in 30-39, 30 is the 'lower limit' and 39 is the 'upper limit'.

(Oranges weight) Class interval	**Tally marks**	**(Number of oranges) Frequency**
30 - 39	卌 |	6
40 - 49	卌 |||	8
50 - 59	卌 ||||	9
60 - 69	卌 |	6
70 - 79	|||	3
80 - 89	卌	5
90 - 99	卌 ||	7
100-109	|||	3
110 - 119	|||	3
	Total	50

Presenting data in this form simplifies and condenses data and enables us to observe certain important features at a glance. This is called a **grouped frequency distribution table**.

We observe that the classes in the table above are non-overlapping i.e. 30-39, 40-49 ... no number is repeating in two class intervals. Such classes are called inclusive classes. Note that we could have made more classes of shorter size, or lesser classes of larger size also. Usually if the raw data is given the range is found (Range = Maximum value – Minimum value). Based on the value of ranges with convenient, class interval length, number of classes are formed. For instance, the intervals could have been 30-35, 36-40 and so on.

Classes	**Class boundaries**
20 - 29	19.5 - 29.5
30 - 39	29.5 - 39.5
40 - 49	39.5 - 49.5
50 - 59	49.5 - 59.5
60 - 69	59.5 - 69.5
70 - 79	69.5 - 79.5
80 - 89	79.5 - 89.5
90 - 99	89.5 - 99.5
100 - 109	99.5-109.5
110 - 119	109.5-119.5
120 - 129	119.5-129.5

Now think if weight of an orange is 39.5 gm. then in which interval will we include it? We cannot include 39.5 either in 30-39 or in 40-49.

In such cases we construct real limits (or boundaries) for every class. Average of upper limit of a class interval and lower limit of the next class interval becomes the upper boundary of the class. The same becomes the lower boundary of the next class interval. Similarly boundaries of all class intervals are calculated.

By assuming a class interval before the first class and next class interval after the last class, we calculate the lower boundary any of the first and upper boundary any of the last class intervals.

Again a problem arises that whether 39.5 has to be included in the class interval 29.5-39.5 or 39.5 - 49.5? Here by convention, if any observation is found to be equivalent to upper boundary of a particular class, then that particular observation is considered under next class, but not that of the particular class.

So 39.5 belongs to 39.5 - 49.5 class interval.

The classes which are in the form of 30-40, 40-50, 50-60, ... are called over lapping classes and called as exclusive classes.

If we observe the boundaries of inclusive classes, they are in the form of exclusive classes. The difference between upper boundary and lower boundary of particular class given the length of class interval. Length of class interval of 90 – 99 is (i.e. 99.5 – 89.5 = 10) 10.

Example-3. The relative humidity (in %) of a certain city for a September month of 30 days was as follows:

98.1	98.6	99.2	90.3	86.5	95.3	92.9	96.3	94.2	95.1
89.2	92.3	97.1	93.5	92.7	95.1	97.2	93.3	95.2	97.3
96.0	92.1	84.9	90.0	95.7	98.3	97.3	96.1	92.1	89

(i) Construct a grouped frequency distirubtion table with classes 84-86, 86,-88 etc.

(ii) What is the range of the data?

Solution : (i) The grouped frequency distribution table is as follows-

Relative humidity	Tally marks	Number of days
84-86	|	1
86-88	|	1
88-90	||	2
90-92	||	2
92-94	卌 ||	7
94-96	卌 |	6
96-98	卌 ||	7
98-100	||||	4

[Note:- 90 comes in interval 90-92 likewise 96 comes in 96-98 class interval]

(ii) Range = maximum value – minimum value

$= 99.2 - 84.9 = 14.3$

EXERCISE - 9.1

1. Write the mark wise frequencies in the following frequency distribution table.

Marks	Up to5	Up to6	Up to7	Up to8	Up to9	Up to10
No of students	5	11	19	31	40	45

2. The blood groups of 36 students of IX class are recorded as follows.

A	O	A	O	A	B	O	A	B	A	B	O	B
O	B	O	O	A	B	O	B	AB	O	A	O	O
O	A	AB	O	A	B	O	A	O	B			

Represent the data in the form of a frequency distribution table. Which is the most common and which is the rarest blood group among these students?

3. Three coins were tossed 30 times simultaneously. Each time the number of heads occurring was noted down as follows;

1	2	3	2	3	1	1	1	0	3	2	1	2
2	1	1	2	3	2	0	3	0	1	2	3	2
2	3	1	1									

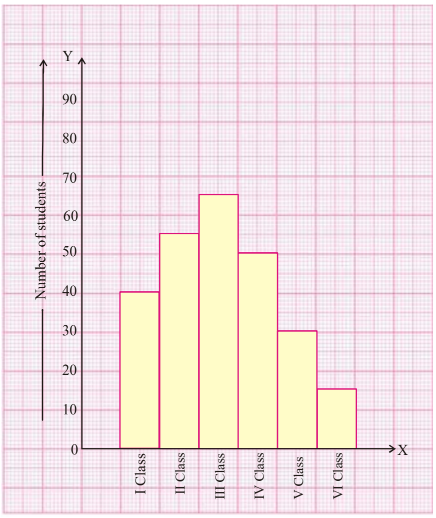
Prepare a frequency distribution table for the data given above.

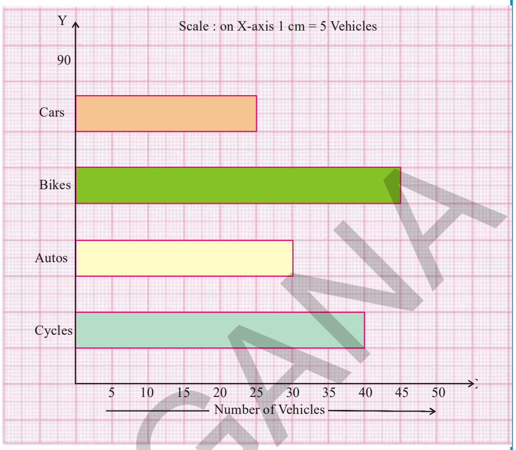
4. A TV channel organized a SMS(Short Message Service) poll on prohibition on smoking, giving options like A – complete prohibition, B – prohibition in public places only, C – not necessary. SMS results in one hour were A B A B C B

A	B	B	A	C	C	B	B	A	B
B	A	B	C	B	A	B	C	B	A
B	B	A	B	B	C	B	A	B	A
B	C	B	B	A	B	C	B	B	A
B	B	A	B	B	A	B	C	B	A
B	B	A	B	C	A	B	B	A	

Represent the above data as grouped frequency distribution table. How many appropriate answers were received? What was the majority of peoples' opinion?

5. Represent the data in the adjacent bar graph as frequency distribution table.





6. Identify the scale used on the axes of the adjacent graph. Write the frequency distribution from it.

7. The marks of 30 students of a class, obtained in a test (out of 75), are given below:

   42, 21, 50, 37, 42, 37, 38, 42, 49, 52, 38, 53, 57, 47, 29

   59, 61, 33, 17, 17, 39, 44, 42, 39, 14, 7, 27, 19, 54, 51.

   Form a frequency table with equal class intervals. (**Hint :** one of them being 0-10)

8. The electricity bills (in rupees) of 25 houses in a locality are given below. Construct a grouped frequency distribution table with a class size of 75.

   170, 212, 252, 225, 310, 712, 412, 425, 322, 325, 192, 198, 230, 320, 412,

   530, 602, 724, 370, 402, 317, 403, 405, 372, 413

9. A company manufactures car batteries of a particular type. The life (in years) of 40 batteries were recorded as follows:

2.6	3.0	3.7	3.2	2.2	4.1	3.5	4.5
3.5	2.3	3.2	3.4	3.8	3.2	4.6	3.7
2.5	4.4	3.4	3.3	2.9	3.0	4.3	2.8
3.5	3.2	3.9	3.2	3.2	3.1	3.7	3.4
4.6	3.8	3.2	2.6	3.5	4.2	2.9	3.6

   Construct a grouped frequency distribution table with exclusive classes for this data, using class intervals of size 0.5 starting from the interval 2 - 2.5.

9.4 Measures of Central Tendency

Consider the following situations:

Case-1 : In a hostel 50 students usually eat 200 idlies in their breakfast. How many more idlies does the mess incharge make if 20 more students joined in the hostel.

Case-2 : Consider the wages of staff at a factory as given in the table. Which salary figure represents the whole staff:

Staff	1	2	3	4	5	6	7	8	9	10
Salary in ₹ (in thousands)	12	14	15	15	15	16	17	18	90	95

Case-3 : The different forms of transport in a city are given below (in percentages). Which is the popular means of transport?

1. Car 15%
2. Train 12%
3. Bus 60%
4. Two wheeler 13%

In the first case, we will usually take an average (mean), and use it to resolve the problem. But if we take average salary in the second case then it would be 30.7 thousands. However, verifying the raw data suggests that this mean value may not be the best way to accurately reflect the typical salary of a worker, as most workers have their salaries between 12 to 18 thousands. So, median (middle value) would be a better measure in this situation. In the third case mode (most frequent) is considered to be a most appropriate option. The nature of the data and its purpose will be the criteria to go for average or median or mode among the measures of central tendency.

Think, Discuss and Write

1. Give 3 situations, where mean, median and mode are separately appropriate and counted.

Considier a situation where fans of two cricketers Raghu and Gautam claim that their star score better than other. They made comparison on the basis of last 5 matches.

Matches		1st	2nd	3rd	4th	5th
Runs made	Raghu	50	50	76	31	100
	Gautam	65	23	100	100	10

Fans of both the players added the runs and calcuated the averages as follows.

$$\text{Raghu's average score} = \frac{307}{5} = 61.4$$

$$\text{Gautam average score} = \frac{298}{5} = 59.6$$

Since Raghu's average score was more than Gautam's, Raghu fan's claimed that Raghu performed better than Gautam, but Gautam fans did not agree. Gautam fan's arranged both their scores in descending order and found the middle score as given below:

Raghu	100	76	50	50	31
Gautam	100	100	65	23	10

Then Gautam fan's said that since his middle-most score is 65, which is higher than Raghu middle-most score, i.e. 50 so his performance should be rated better.'

But we may say that Gautam made two centuries in 5 matches and so he may be better.

Now, to settle the dispute between Raghu's and Gautam's fans, let us see the three measures adopted here to make their point.

The average score they used first is the mean. The 'middle' score they used in the argument is the Median. Mode is also a measure to compare the performance by considering the scores repeated many times. Mode score of Raghu is 50. Mode score of Gautam is 100. Of all these three measures which one is appropriate in this context?

when we use arithmetic mean, median, mode?

Now let us first understand mean in details.

9.4.1 Arithmetic Mean / Average

Mean is the 'sum of observations of a data divided by the number of observations'. We have already discussed about computing arithmetic mean for a raw data.

$$\text{Mean } \bar{x} = \frac{\text{Sum of observations}}{\text{Number of observations}} \text{ or } \bar{x} = \frac{\Sigma x_i}{n}$$

9.4.1.1 Mean of Raw Data

Example-4. Rain fall of a place in a week is 4cm, 5cm, 12cm, 3cm, 6cm, 8cm, 0.5cm. Find the average rainfall per day.

Solution : The average rainfall per day is the arithmetic mean of the above observations.

Given rainfall through a week are 4cm, 5cm, 12cm, 3cm, 6cm, 8cm, 0.5cm.

Number of observations $(n) = 7$

$$\text{Mean } \bar{x} = \frac{\sum x_i}{n} = \frac{x_1 + x_2 + x_3 + ...x_n}{n}, \text{ Where } x_1, x_2 \text{ ..... } x_n \text{ are } n \text{ observation}$$

$$\text{and } \bar{x} \text{ is their mean} = \frac{4+5+12+3+6+8+0.5}{7} = \frac{38.5}{7} = 5.5 \text{ cm.}$$

Example-5. If the mean of 10, 12, 18, 13, P and 17 is 15, find the value of P.

Solution : We know that Mean $\bar{x} = \frac{\sum x_i}{n}$

$$15 = \frac{10+12+18+13+P+17}{6}$$

$$90 = 70 + P$$

$$P = 20.$$

9.4.1.2 Mean of Ungrouped frequency distribution

Consider this example; Weights of 40 students in a class are given in the following frequency distribution table.

Weights in kg (x)	30	32	33	35	37	41
No of students (f)	5	9	15	6	3	2

Find the average (mean) weight of 40 students.

From the table we can see that 5 students weigh 30 kg., each. So sum of their weights is $5 \times 30 = 150$ kg. Similarly we can find out the sum of weights with each frequency and then their total. Sum of the frequencies gives the number of observations in the data.

$$\text{Mean } (\bar{x}) = \frac{\text{Sum of all the observations}}{\text{Total number of observations}}$$

$$\text{So Mean} = \frac{5\times30+9\times32+15\times33+6\times35+3\times37+2\times41}{5+9+15+6+3+2} = \frac{1336}{40} = 33.40 \text{ kg.}$$

If observations are $x_1, x_2, x_3, x_4, x_5, x_6$ and corresponding frequencies are $f_1, f_2, f_3, f_4, f_5, f_6$ then we may write the above expression as

$$\text{Mean } \bar{x} = \frac{f_1x_1 + f_2x_2 + f_3x_3 + f_4x_4 + f_5x_5 + f_6x_6}{f_1 + f_2 + f_3 + f_4 + f_5 + f_6} = \frac{\sum f_i x_i}{\sum f_i}$$

$$\therefore \bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

Example-6. Find the mean of the following data.

x	5	10	15	20	25
f	3	10	25	7	5

Solution :

Step-1 : Calculate $f_i \times x_i$ of each row

Step-2 : Find the sum of frequencies (Σf_i)

and sum of the $f_i \times x_i$ ($\Sigma f_i x_i$)

Step-3 : Calculate $\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{755}{50} = 15.1$

x_i	f_i	$f_i x_i$
5	3	15
10	10	100
15	25	375
20	7	140
25	5	125
	$\Sigma f_i = 50$	$\Sigma f_i x_i = 755$

Example-7. If the mean of the following data is 7.5 , then find the value of 'A'.

Marks	5	6	7	8	9	10
No. of Students	3	10	17	A	8	4

Solution :

Sum of frequencies $(\Sigma f_i) = 42 + \text{A}$

Sum of the $f_i \times x_i$ $(\Sigma f_i x_i) = 306 + 8\text{A}$

$$\text{Mean } \bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i}$$

Given Arithmetic Mean $= 7.5$

$$\text{So, } \quad 7.5 = \frac{306 + 8\text{A}}{42 + \text{A}}$$

$$306 + 8\text{A} = 315 + 7.5\,\text{A}$$

Marks (x_i)	No. of Students (f_i)	$f_i x_i$
5	3	15
6	10	60
7	17	119
8	A	8A
9	8	72
10	4	40
	42+A	306+8A

$$8A - 7.5\,A = 315 - 306$$

$$0.5\,A = 9$$

$$A = 18$$

9.4.1.3 Mean of ungrouped frequency Distribution by Deviation method

Example-8. Find the arithmetic mean of the following data:

x	10	12	14	16	18	20	22
f	4	5	8	10	7	4	2

Solution :

(i) Simple Method

Thus in the case of ungrouped frequency distribution, you can use the formula,

$$\bar{x} = \frac{\sum_{i=1}^{7} f_i x_i}{\sum_{i=1}^{7} f_i} = \frac{622}{40} = 15.55$$

x_i	f_i	$f_i x_i$
10	4	40
12	5	60
14	8	112
16	10	160
18	7	126
20	4	80
22	2	44
	$\sum_{i=1}^{7} f_i = 40$	$\sum_{i=1}^{7} f_i x_i = 622$

(ii) Deviation Method

In this method we assume one of the observations which is convenient as assumed mean. Suppose we assume '16' as a mean, be A = 16 the deviation of other observations from the assumed mean are given in table.

x_i	f_i	$d_i = x_i - A$	$f_i d_i$
10	4	–6	–24
12	5	–4	–20
14	8	–2	–16
16 A	10	0	0
18	7	+2	+14
20	4	+4	+16
22	2	+6	+12
	40		–60+42=–18

Sum of frequencies = 40

Sum of the $f_i \times d_i$ products $= -60 + 42$

$$\Sigma f_i d_i = -18$$

Mean $\bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i} = 16 + \left(\frac{-18}{40}\right)$

$= 16 - 0.45$

$= 15.55$

9.4.2 Median

Median is the middle observation of a given raw data, when it is arranged in an order (ascending/descending). It divides the data into two groups of equal number, one part comprising all values greater than median and the other part comprising values less than median.

We have already discussed in the earlier classes that median of a raw data with observations, arranged in order is calculated as follows.

When the data has 'n' number of observations and if 'n' is odd, median is $\left(\frac{n+1}{2}\right)^{th}$ observation.

When n is even, median is the average of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations

TRY THESE

1. Find the median of the scores 75, 21, 56, 36, 81, 05, 42
2. Median of a data, arranged in ascending order 7, 10, 15, x, y, 27, 30 is 17 and when one more observation 50 is added to the data, the median has become 18 Find x and y.

9.4.2.1 Median of a frequency distribution

Let us now discuss the method of finding the median for a data of weighted observations consider the monthly wages of 100 employees of a company given in the following table.

Wages (in ₹)	7500	8000	8500	9000	9500	10000	11000
No. of employees	4	18	30	20	15	8	5

How to find the median of the given data? First arrange the observations given either in ascending or descending order. Then write the corresponding frequencies in the table and calculate less than cumulative frequencies. The cumulative frequency upto a particular observation is the progressive sum of frequencies upto that particular observation. The cumulative frequency of the first observation in the data is the frequency of that observation itself.

Wages (x)	No.of employees (f)	Cumulative frequency (cf)
7500	4	4
8000	18	22
8500	30	52
9000	20	72
9500	15	87
10000	8	95
11000	5	100
	100	

Find $\frac{N}{2}$ and identify the median class, whose cumulative frequencies just exceeds $\frac{N}{2}$, where N is sum of the frequencies.

Here N= 100 even so find $\left(\frac{N}{2}\right)^{th}$ and $\left(\frac{N}{2}+1\right)^{th}$ observations which are 50 and 51 respectively.

From the table corresponding values of 50^{th} and 51^{st} observations is the same, falls in the wages of 8500. So the median class of this distribution is 8500.

Try These

1. Find the median marks in the data.

Marks	15	20	10	25	5
No of students	10	8	6	4	1

2. In finding the median, the given data must be written in order. Why?

9.4.3 Mode

Mode is the value of the observation which occurs most frequently, i.e., an observation with the maximum frequency is called mode.

Example-9. The following numbers are the sizes of shoes sold by a shop in a particular day. Find the mode.

6, 7, 8, 9, 10, 6, 7, 10, 7, 6, 7, 9, 7, 6.

Solution : First we have to arrange the observations in order 6, 6, 6, 6, 7, 7, 7, 7, 7, 8, 9, 9, 10, 10 to make frequency distribution table

Size	6	7	8	9	10
No. of shoes sold	4	5	1	2	2

Here No. 7 occurred most frequently. i.e, 5 times.

$\therefore$ Mode (size of the shoes) of the given data is 7. This indicates the shoes of size No. '7' is a most selling item.

Think and Discuss

1. Classify your class mates according to their heights and find the mode of it.
2. If shopkeeper has to place a order for shoes, which number shoes should he order more?

Example-10. Test scores out of 100 for a class of 20 students are as follows:

93, 84, 97, 98, 100, 78, 86, 100, 85, 92, 55, 91, 90, 75, 94, 83, 60, 81, 95

(a) Make a frequency table taking class interval as 91-100, 81-90, ...

(b) Find the modal class. (The "Modal class" is the class containing the greatest frequency).

(c) find the interval that contains the median.

Solution :

(a)

Test Scores	Frequency	Greater than Cumulative frequency
91-100	9	20
81-90	6	11
71-80	3	5
61-70	0	2
51-60	2	2
Total	**20**	

(b) 91-100 is the modal class. This class has the maximum frequency.

(c) The middle of 20 is 10. If I count from the top, 10 will fall in the class interval 81-90. If I count from the bottom and go up, 10 will fall in the class interval 81-90. The class interval that contains the median is 81-90.

9.5 Deviation in values of central tendency

What will happen to the measures of central tendency if we add the same amount to all data values, or multiply each data value by the same amount.

Let us observe the following table

Particular	Data	Mean	Mode	Median
Original Data Set	6, 7, 8, 10, 12, 14, 14, 15, 16, 20	12.2	14	13
Add 3 to each data value	9, 10, 11, 13, 15, 17, 17, 18, 19, 23	15.2	17	16
Multiply 2 times each data value	12, 14, 16, 20, 24, 28, 28, 30, 32, 40	24.4	28	26

After observing the table, we can say

When added : Since all values are shifted by the same amount, the measures of central tendency are all shifted by the same amount. If 3 is added to each data value, the mean, mode and median will also increase by 3.

When multiplied : Since all values are affected by the same multiplicative values, the measures of central tendency will also be affected similarly. If each observation is multiplied by 2, the mean, mode and median will also be multiplied by 2.

EXERCISE - 9.2

1. Weights of parcels in a transport office are given below. Find the mean weight of the parcels.

Weight (kg)	50	65	75	90	110	120
No of parcels	25	34	38	40	47	16

2. Number of families in a village in correspondence with the number of children are given below: Find the mean number of children per family.

No of children	0	1	2	3	4	5
No of families	11	25	32	10	5	1

3. If the mean of the following frequency distribution is 7.2 find value of 'K'.

x	2	4	6	8	10	12
f	4	7	10	16	K	3

4. Number of villages with respect to their population as per India census 2011 are given below (in thousands). Find the average population in each village.

Population (in thousands)	12	5	30	20	15	8
No. of Villages	20	15	32	35	36	7

5. AFLATOUN social and financial educational program intiated savings program among the high school children in Hyderabad district. Mandal wise savings in a month are given in the following table.

Mandal	No. of schools	Total amount saved (in rupees)
Amberpet	6	2154
Thirumalgiri	6	2478
Saidabad	5	975
Khairathabad	4	912
Secundrabad	3	600
Bahadurpura	9	7533

Find arithmetic mean of school wise savings in each mandal. Also find the arithmetic mean of saving of all schools.

6. The heights of boys and girls of IX class of a school are given below.

Height (cm)	135	140	147	152	155	160
Boys	2	5	12	10	7	1
Girls	1	2	10	5	6	5

Compare the heights of the boys and girls

[**Hint** : Find median heights of boys and girls]

7. Centuries scored and number of cricketers in the world are given below.

No. of centuries	5	10	15	20	25
No. of cricketers	56	23	39	13	8

Find the mean, median and mode of the given data.

8. On the occasion of New year's day a sweet stall prepared sweet packets. Number of sweet packets and cost of each packet are given as follows.

Cost of packet (in ₹)	₹25	₹50	₹75	₹100	₹125	₹ 150
No of packets	20	36	32	29	22	11

Find the mean, median and mode of the data.

9. The mean (average) weight of three students is 40 kg. One of the students Ranga weighs 46 kg. The other two students, Rahim and Reshma have the same weight. Find Rahim's weight.

10. The donations given to an orphanage home by the students of different classes of a secondary school are given below. (in Rs.)

Class	Donation by each student (in ₹)	No. of students donated
VI	5	15
VII	7	15
VIII	10	20
IX	15	16
X	20	14

Find the mean, median and mode of the data.

11. There are four unknown numbers. The mean of the first two numbers is 4 and the mean of the first three is 9. The mean of all four number is 15, if one of the four number is 2 find the other numbers.

WHAT WE HAVE DISCUSSED?

W9Z8U7

- Representation of the data with actual observations with frequencies in a table is called '**Ungrouped Frequency Distribution Table**' or '**Table of Weighted Observations**'
- Representation of a large data in the form of a frequency distribution table enables us to view the data at a glance, to find the range easily, to find which observation is repeating for how many times, to analyse and to interpret the data easily.
- A measure of central tendency is a typical value of the data around which other observations congregate.
- Types of measure of central tendency : Mean, Mode, Median.
- Mean is the sum observations of a data divided by the number of observations.

$$\text{Mean} = \frac{\text{Sum of observations}}{\text{Number of observations}} \quad \text{or} \quad \bar{x} = \frac{\Sigma x_i}{n}$$

- For a ungrouped frequency distribution arithmetic mean $\bar{x} = \dfrac{\Sigma f_i x_i}{\Sigma f_i}$.

- By deviation method, arithmetic mean $= A + \frac{\Sigma f_i d_i}{\Sigma f_i}$ where A is assumed mean and Σf_i is the sum of frequencies and $\Sigma f_i d_i$ is the sum of product of frequency and deviations.
- Median is the middle observation of a data, when arranged in order (ascending /descending).
- When number of observations 'n' is odd, median is $\left(\frac{n+1}{2}\right)^{th}$ observation.
- When number of observations 'n' is even, median is the average of $\left(\frac{n}{2}\right)^{th}$ and $\left(\frac{n}{2}+1\right)^{th}$ observations
- Median divides the data into two groups of equal number, one part comprising all values greater and the other comprising all values less than median.
- Mode is the value of the observation which occurs most frequently, i.e., an observation with the maximum frequency is called mode.

Brain teaser

1. In a row of students, Gopi is the 7th boy from left and Shankar is the 5th boy from the right. If they exchange their seats, Gopi is the 8th boy from the right. How many students are there in the row?
2. A boy Chaitanya carved his name on the bark of a tree of height 4.5m at a height of 1.5 m tall. After 10 years the tree attains a height of a 6.75 m from the ground at what height from the ground will Chaitanya's name can be located now?

   Give reason to your answer.

Chapter

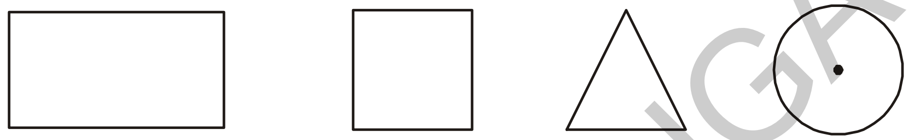
10 Surface Areas and Volumes

M1N4Y1

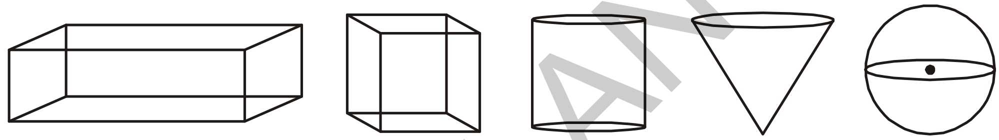
10.1 Introduction

Observe the following figures

(a)



(b)

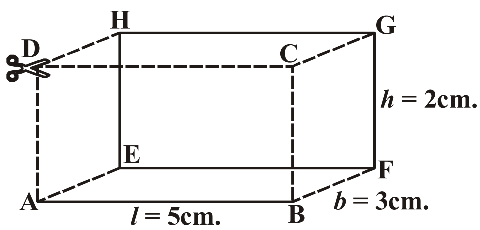


Have you noticed any differences between the figures of group (a) and (b)?

From the above, figures of group (a) can be drawn easily on our note books. These figures have length and breadth only and are named as two dimensional figures or 2-D objects. In group (b) the figures, which have length, breadth and height are called as three dimensional figures or 3-D objects. These are called solid figures. Usually we see solid figures in our surroundings. You have learned about plane figures and their areas. We shall now learn to find the surface areas and volumes of 3-dimensional objects such as cylinders, cones and spheres.

10.2 Surface Area of Cuboid

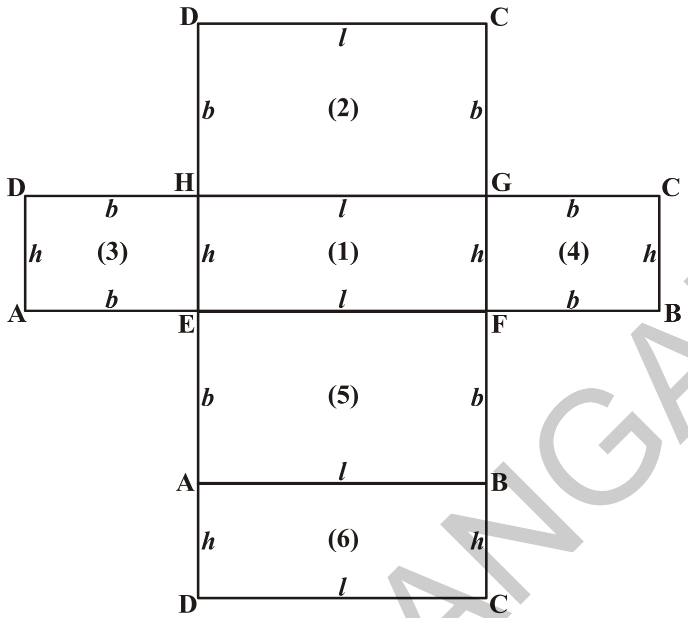
Observe the cuboid and find how many faces it has? How many corners and how many edges it has? Which pair of faces are equal in size? Do you get any idea to find the surface area of the cuboid?



Now let us find the total surface area of a cuboid.

In the above figure length (l) = 5 cm; breadth (b) = 3 cm; height (h) = 2 cm

If we cut and open the given cuboid along CD, ADHE and BCGF. The figure we obtained is shown below:



This shows that the surface area of a cuboid is made up of six rectangles of three identical pairs of rectangles. To get the total surface area of cuboid, we have to add the areas of all six rectangular faces. The sum of these areas gives the total surface area of a cuboid.

Area of the rectangle EFGH = $l \times h = lh$ .....(1)

Area of the rectangle HGCD = $l \times b = lb$ .....(2)

Area of the rectangle AEHD = $b \times h = bh$ .....(3)

Area of the rectangle FBCG = $b \times h = bh$ .....(4)

Area of the rectangle ABFE = $l \times b = lb$ .....(5)

Area of the rectangle DCBA = $l \times h = lh$ .....(6)

On adding the above areas, we get the surface area of cuboid.

Surface Area of a cuboid = Areas of (1) + (2) + (3) + (4) + (5) + (6)

$= lh + lb + bh + bh + lb + lh$

$= 2\ lb + 2lh + 2bh$

$= 2(lb + bh + lh)$

(1), (3), (4), (6) are lateral surfaces of the cuboid

Lateral Surface Area of a cuboid = Area of (1) + (3) + (4) + (6)

$= lh + bh + bh + lh$

$= 2lh + 2bh$

$= 2h\ (l + b)$

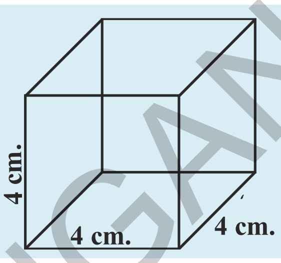
Now let us find the surface areas of cuboid for the above figure. Thus total surface area is 62 cm.^{2} and lateral surface area is 32 cm.^{2}.

Try This

Take a cube of edge 'l' cm. and cut it as we did in the previous activity and find total surface area and lateral surface area of cube.

Do This

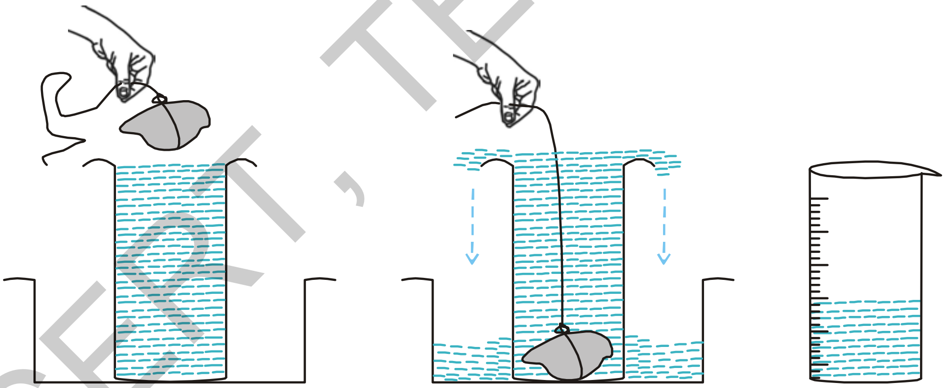
1. Find the total Surface area and lateral surface area of the Cube with side 4 cm. (By using the formulae deduced above)
2. Each edge of a cube is increased by 50%. Find the percentage increase in the total surface area.



10.2.1 Volume

To recall the concept of volume, Let us do the following activity.

Take a glass jar, place it in a container. Fill the glass jar with water up to its brim. Slowly drop a solid object (a stone) in it. Some of the water from the jar will overflow into the container. Take the overflowed water into measuring jar. It gives an idea of space occupied by a solid object called volume.



Every object occupies some space, the space occupied by an object is called its volume. Volume is measured in cubic units.

10.2.2 Capacity of the container

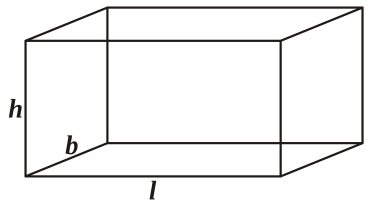
If the object is hollow, then interior is empty and it can be filled with air or any other liquid, that will take the shape of its container. Volume of the substance that can fill the interior is called the capacity of the container.

Volume of a Cuboid : Cut some rectangles from a cardboard of same dimensions and arrange them one over other. What do you say about the shape so formed?

The shape is a cuboid.

Now let us find volume of a cuboid.

Its length is equal to the length of the rectangle, and breadth is equal to the breadth of the rectangle.



The height up to which the rectangles are stacked is the height of the cuboid is 'h'

Space occupied by the cuboid = Area of plane region occupied by rectangle × height

Volume of the cuboid = $l\, b \times h = l\, bh$

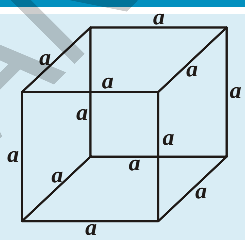
$\therefore$ Volume of the cuboid = $l\, bh$

Where l, b, h are length, breadth and height of the cuboid.

Try These

(a) Find the volume of a cube whose edge is 'a' units.

(b) Find the edge of a cube whose volume is 1000 cm^3.



We know that cuboid and cube are the solids. Do we call them as right prisms? You have observed that cuboid and cube are also called right prisms as their lateral faces are rectangle and perpendicular to base.

We know that the volume of a cuboid is the product of the area of its base and height.

Remember that volume of the cuboid = Area of base × height

$= lb \times h$

$= lbh$

In cube $= l = b = h = s$ (All the dimensions are same)

volume of the cube $= s^2 \times s$

$= s^3$

Hence volume of a cuboid should hold good for all right prisms.

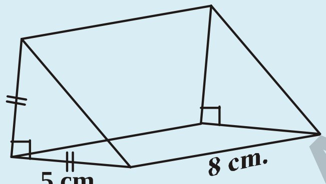
Hence volume of right prism = Area of the base × height

In particular, if the base of a right prism is an equilateral triangle its volume is $\frac{\sqrt{3}}{4} a^2 \times$ h cu.units.

Where, 'a' is the length of each side of the base and 'h' is the height of the prim.

Do These

1. Find the volume of cuboid if $l = 12$ cm., $b = 10$ cm. and $h = 8$ cm.
2. Find the volume of cube, if its edge is 10 cm.
3. Find the volume of isosceles right angled triangular prism in (fig.).



(Fig.)

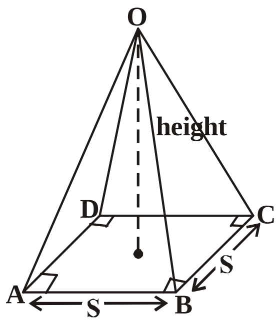
Like the prism, the pyramid is also a three dimentional solid figure. This figure has fascinated human beings from the ancient times. You might have read about pyramids of Egypt, which are, one of the seven wonders of the world. They are the remarkable examples of pyramids on square bases. How are they built? It is a mystery. No one really knows that how these massive structures were built.

Can you draw the shape of a pyramid?

What is the difference you have observed between the prism and pyramid?

What do we call a pyramid of square base?

Here OABCD is a square pyramid of side 'S' units and height 'h' units.

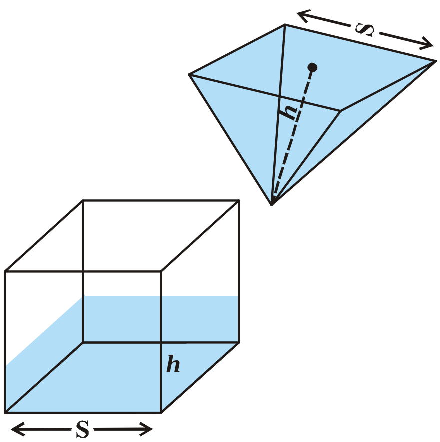


Can you guess the volume of a square pyramid in terms of volume of cube if their bases and height are equal?

Activity

Take the square pyramid and cube containers of same base and with equal heights.

Fill the pyramid with a liquid and pour into the cube (prism) completely. How many times it takes to fill the cube? On examining it takes 3 times.



Thus volume of pyramid

$$= \frac{1}{3} \text{ of the volume of right prism.}$$

$$= \frac{1}{3} \times \text{Area of the base} \times \text{height.}$$

Note : A Right prism has bases perpendicular to the lateral edges and all lateral faces are rectangles.

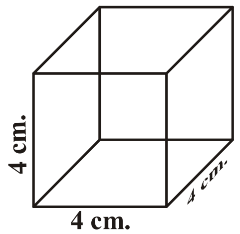
Do These

1. Find the volume of a pyramid whose square base is 10 cm. and height 8 cm.
2. The volume of cube is 1728 cubic cm. Find the volume of square pyramid of the same height.

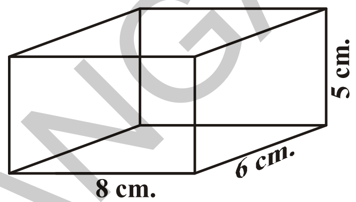
Exercise - 10.1

1. Find the later surface area and total surface area of the following right prisms.

(i)



(ii)



2. The total surface area of a cube is 1350 sq.m. Find its volume.
3. Find the area of four walls of a room (Assume that there are no doors or windows) if its length is 12 m., breadth is 10 m. and height is 7.5 m.
4. The volume of a cuboid is 1200 cm^3. The length is 15 cm. and breadth is 10 cm. Find its height.
5. How does the total surface area of a box change if

(i) Each dimension is doubled? (ii) Each dimension is tripled?

Express in words. Can you find the total surface area of the box if each dimension is raised to *n* times?

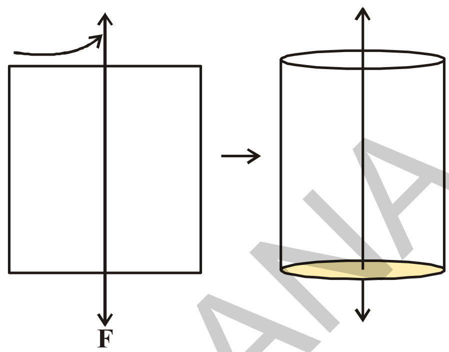
6. The base of a prism is triangular in shape with sides 3 cm., 4 cm. and 5 cm. Find the volume of the prism if its height is 10 cm.
7. A regular square pyramid is 3 m. height and the perimeter of its base is 16 m. Find the volume of the pyramid.
8. An Olympic swimming pool is in the shape of a cuboid of dimensions 50 m. long and 25 m. wide. If it is 3 m. deep throughout, how many liters of water does it hold? (1 cu.m = 1000 liters)

Activity

Cut out a rectangular sheet of paper. Paste a thick string along the line as shown in the figure. Hold the string with your hands on either sides of the rectangle and rotate the rectangle sheet about the string as fast as you can.

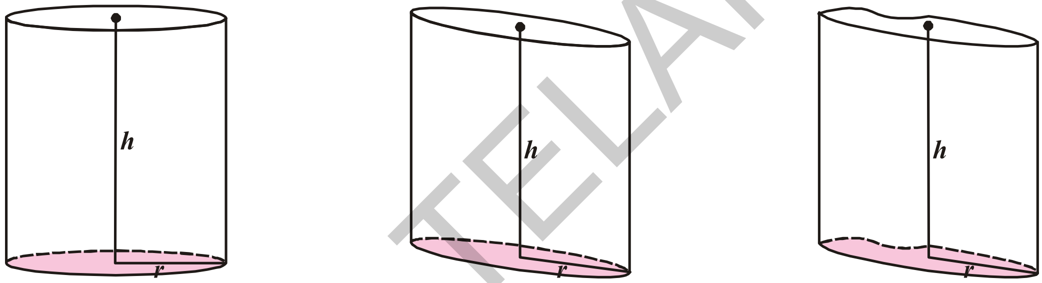
Do you recognize the shape that the rotating rectangle is forming ?

Does it remind you the shape of a cylinder ?



10.3 Right Circular Cylinder

Observe the following cylinders:



(i) (ii) (iii)

(i) What similiarties you have observed in the 3 figures?

(ii) What differences you have observed between in the 3 figures?

(iii) In which figure, the line segment is perpendicular to the base?

Every cylinder is made up of one curved surface and with two congruent circular faces on both ends. If the line segment joining the centre of circular faces, is perpendicular to its base, such a cylinder is called right circular cylinder or right cylinder.

Find out which is right circular cylinder in the above figures? Which are not? Give reasons.

Let us do an activity to generate a cylinder

10.3.1 Curved Surface area of a cylinder

Take a right circular cylinder made up of cardboard. Cut the curved face vertically and unfold it. While unfolding cylinder, observe its transformation of its height and the circular base. After unfolding the cylinder what shape do you find?

You will find it is in rectangular shape. The area of rectangle is equal to the curved surface area of cylinder. Its height is equal to the breadth of the rectangle, and the circumference of the base is equal to the length of the rectangle.

Height of cylinder = breadth of rectangle ($h = b$)

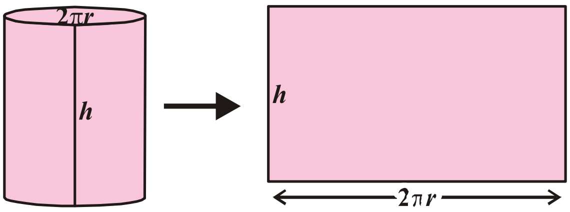
Circumferance of base of cylinder with radius 'r' = length of the rectangle ($2\pi r = l$)

Curved surface area of the cylinder = Area of the rectangle

$= \text{length} \times \text{breadth}$

$= 2\pi r \times h$

$= 2\pi rh$



Therefore, Curved surface area of a cylinder = $2\pi rh$

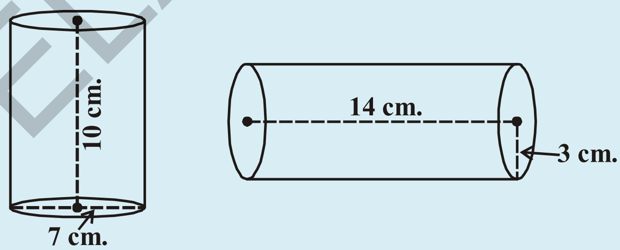
Do This

Find curved surface area of each of following cylinders

(i) $r = x$ cm., $h = y$ cm.

(ii) $d = 7$ cm., $h = 10$ cm.

(iii) $r = 3$ cm., $h = 14$ cm.



10.3.2 Total Surface area of a Cylinder

Observe the adjacent figure.

Do you find that it is a right circular cylinder? What surfaces you have to add to get its total surface area? They are the curved surface area and area of two circular faces.

Now the total surface area of a cylinder

= Curved surface area + Area of top + Area of base

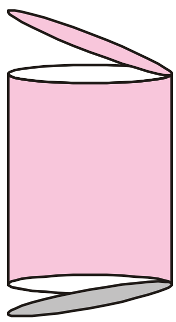
$= 2\pi rh + \pi r^2 + \pi r^2$

$= 2\pi rh + 2\pi r^2$

$= 2\pi r\,(h + r)$

$= 2\pi r\,(r + h)$

$\therefore$ The total surface area of a cylinder $= 2\pi r\,(r + h)$

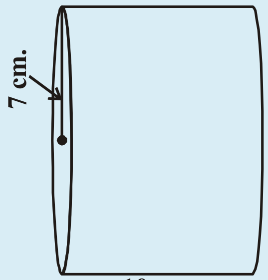


Where 'r' is the radius of the cylinder and 'h' is its height.

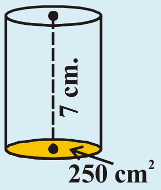
Do These

Find the Total surface area of each of the following cylinders.

(i)



(ii)

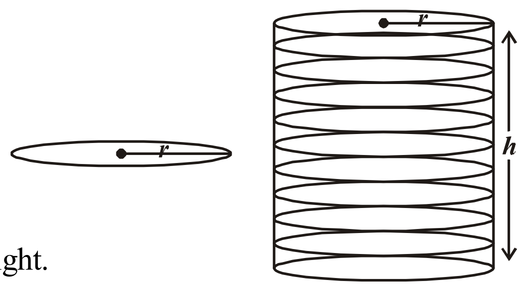


10.3.3 Volume of a cylinder

Take circles with equal radii and arrange one over the other.

Do this activity and find whether it form a cylinder or not?

In the figure 'r' is the radius of the circle, and the 'h' is the height up to which the circles are stacked.



Volume of a cylinder $= \pi r^2 \times$ height

$= \pi r^2 \times h$

$= \pi r^2 h$

So volume of a cylinder = $\pi r^2 h$

Where 'r' is the radius of cylinder and 'h' is its height.

Example-1. A Rectangular paper of width 14 cm is folded along its width and a cylinder of radius 20 cm is formed. Find the volume of the cylinder (Fig 1) ? $\left(\text{Take } \pi = \frac{22}{7}\right)$

Solution : A cylinder is formed by rolling a rectangle about its width. Hence the width of the paper becomes height of cylinder and radius of the cylinder is 20 cm.

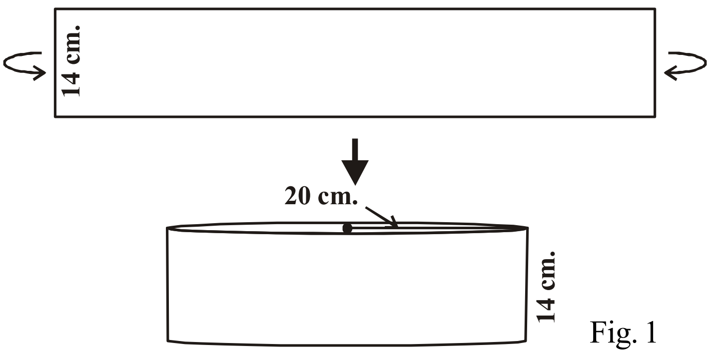


Fig. 1

Height of the cylinder = h = 14 cm.

radius (r) = 20 cm.

Volume of the cylinder V = $\pi r^2 h$

$$= \frac{22}{7} \times 20 \times 20 \times 14$$

$$= 17600 \text{ cm}^3.$$

Hence the volume of the cylinder is 17600 cm^3.

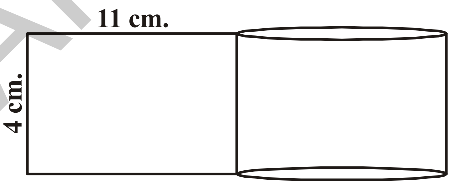
Example-2. A Rectangular piece of paper 11 cm × 4 cm is folded without overlapping to make a cylinder of height 4 cm. Find the volume of the cylinder.

Solution : Length of the paper becomes the circumference of the base of the cylinder and width becomes height.

Let radius of the cylinder = r and height = h

Circumference of the base of the cylinder = $2\pi r = 11$ cm.

$$2 \times \frac{22}{7} \times r = 11$$



$$\therefore \quad r = \frac{7}{4} \text{ cm.}$$

$$h = 4 \text{ cm}$$

Volume of the cylinder $(V) = \pi r^2 h$

$$= \frac{22}{7} \times \frac{7}{4} \times \frac{7}{4} \times 4$$

$$= 38.5 \text{ cm}^3.$$

Example-3. A rectangular sheet of paper 44 cm × 18 cm is rolled along the length to form a cylinder. Assuming that the cylinder is solid (Completely filled), find its radius and the total surface area.

Solution : Height of the cylinder = 18 cm

Circumference of base of cylinder = 44 cm

$$2\pi r = 44 \text{ cm}$$

$$r = \frac{44}{2 \times \pi} = \frac{44 \times 7}{2 \times 22} = 7\text{cm.}$$

Total surface area $= 2\pi r\,(r + \text{h})$

$$= 2\times\frac{22}{7}\times 7(7+18)$$

$$= 1100 \text{ cm}^2.$$

Example-4. Circular discs 5 mm thickness, are placed one above the other to form a cylinder of curved surface area 462 cm^2. Find the number of discs, if the radius is 3.5 cm.

Solution : Thickness of disc $= 5 \text{ mm} = \frac{5}{10}\text{cm} = 0.5\,\text{cm}$

Radius of disc = 3.5 cm.

Curved surface area of cylinder = 462 cm^2.

$\therefore\ 2\pi\text{rh} = 462$ ..... (i)

Let the no of discs be x

$\therefore$ Height of cylinder = h = Thickness of disc × no of discs

$= 0.5\ x$

$\therefore\ 2\ \pi\text{rh} = 2\times\frac{22}{7}\times 3.5\times 0.5x$ ..... (ii)

From (i) are (ii) we get

$$2\times\frac{22}{7}\times 3.5\times 0.5x = 462$$

$$\therefore\ x = \frac{462\times 7}{2\times 22\times 3.5\times 0.5} = 42 \text{ discs}$$

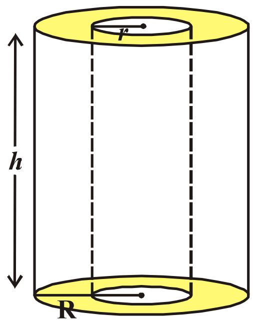
Example-5. A hollow cylinder having external radius 8 cm and height 10 cm has a total surface area of $338\,\pi\ \text{cm}^2$. Find the thickness of the hollow metallic cylinder.

Solution : External radius = R = 8 cm

Internal radius = r

Height = 10 cm

TSA $= 338\pi\ \text{cm}^2$.



But TSA = Area of external cylinder (CSA)

+ Area of internal cylinder (CSA)

+ 2 × Area of base (ring)

$= 2\pi Rh + 2\pi rh + 2\pi (R^2 - r^2)$

$= 2\pi (Rh + rh + R^2 - r^2)$

$\therefore\ 2\pi (Rh + rh + R^2 - r^2) = 338\,\pi$

$Rh + rh + R^2 - r^2 = 169$

$\Rightarrow\ (10 \times 8) + (r \times 10) + 8^2 - r^2 = 169$

$\Rightarrow\ r^2 - 10r + 25 = 0$

$\Rightarrow\ (r - 5)^2 = 0$

$\therefore\ r = 5$

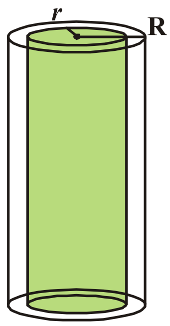
$\therefore$ Thickness of metal = R – r = (8 – 5) cm = 3 cm.

Try These

1. If the radius of a cylinder is doubled keeping its lateral surface area the same, then what is its height ?
2. A hot water system (Geyser) consists of a cylindrical pipe of length 14 m and diameter 5 cm. Find the total radiating surface of hot water system.

Exercise - 10.2

1. A closed cylindrical tank of height 1.4 m. and radius of the base is 56 cm. is made up of a thick metal sheet. How much metal sheet is required (Express in square meters)
2. The volume of a cylinder is 308 cm.^{3}. Its height is 8 cm. Find its lateral surface area and total surface area.
3. A metal cuboid of dimension 22 cm. × 15 cm. × 7.5 cm. was melted and cast into a cylinder of height 14 cm. What is its radius?
4. An overhead water tanker is in the shape of a cylinder has capacity of 61.6 cu.mts. The diameter of the tank is 5.6 m. Find the height of the tank.
5. A metal pipe is 77 cm. long. The inner diameter of a cross section is 4 cm., the outer diameter being 4.4 cm. (see figure) Find its
   (i) inner curved surface area
   (ii) outer curved surface area
   (iii) Total surface area.



6. A cylindrical piller has a diameter of 56 cm and is of 35 m high. There are 16 pillars around the building. Find the cost of painting the curved surface area of all the pillars at the rate of ₹5.50 per 1 m^2.

7. The diameter of a roller is 84 cm and its length is 120 cm. It takes 500 complete revolutions to roll once over the play ground to level. Find the area of the play ground in m^2.

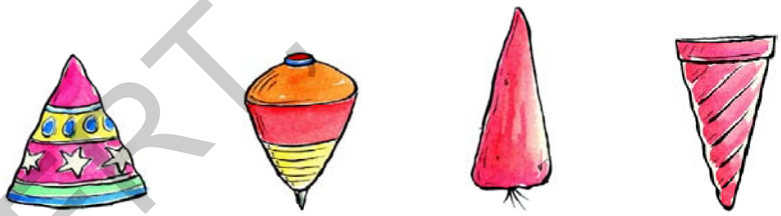
8. The inner diameter of a circular well is 3.5 m. It is 10 m deep. Find
   (i) its inner curved surface area
   (ii) The cost of plastering this curved surface at the rate of Rs. 40 per m^2.

9. Find
   (i) The total surface area of a closed cylindrical petrol storage tank whose diameter 4.2 m. and height 4.5 m.
   (ii) How much steel sheet was actually used, if $\frac{1}{12}$ of the steel was wasted in making the tank.

10. A one side open cylinderical drum has inner radius 28 cm. and height 2.1 m. How much water you can store in the drum. Express in litres. (1 litre = 1000 cc.)

11. The curved surface area of the cylinder is 1760 $cm.^2$ and its volume is 12320 cm^3. Find its height.

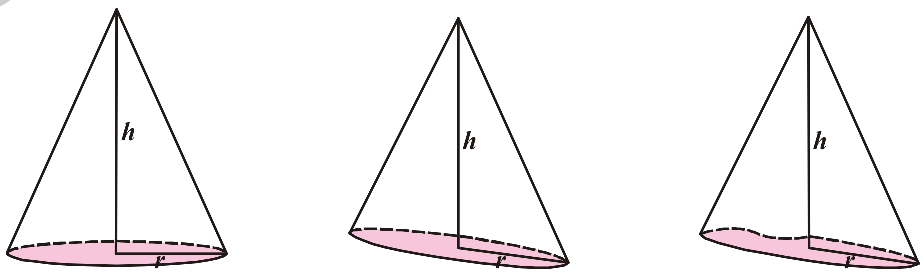
10.4 Right Circular Cone



Observe the above figures and which solid shape they resemble?

These are in the shape of a cone.

Observe the following cones:



(i) (ii) (iii)

(i) What common properties do you find among these cones?

(ii) What difference do you notice among them?

In fig.(i), lateral surface is curved and base is circle. The line segment joining the vertex of the cone and the centre of the circular base (vertical height) is perpendicular to the radius of the base. This type of cone is called Right Circular Cone.

In fig.(ii) although it has circular base, but its vertical height is not perpendicular to the radius of the cone.

Such type of cones are not right circular cones.

In the fig. (iii) although the vertical height is perpendicular to the base, but the base is not in circular shape.

Therefore, this cone is not a right circular cone.

10.4.1 Slant Height of the Cone

In the adjacent figure (cone), $\overline{AO}$ is perpendicular to $\overline{OB}$

ΔAOB is a right angled triangle.

$\overline{AO}$ is the height of the cone (h) and $\overline{OB}$ is equal to the radius of the cone (r)

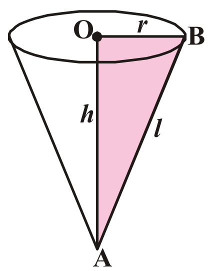
From ΔAOB

$AB^2 = AO^2 + OB^2$

$AB^2 = h^2 + r^2$ (AB is called slant height $= l$)

$l^2 = h^2 + r^2$

$l = \sqrt{h^2 + r^2}$



Activity

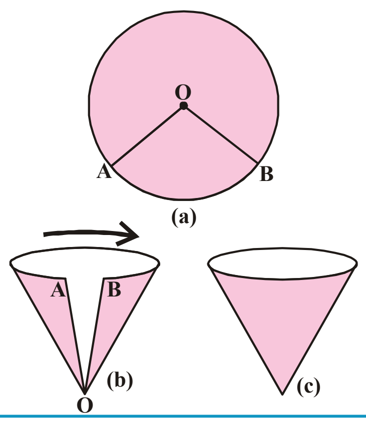
Making a cone from a sector

Follow the instructions and do as shown in the figure.

(i) Draw a circle on a thick paper Fig(a)

(ii) Cut a sector AOB from it Fig(b).

(iii) Fold the ends A, B nearer to each other slowly and join A, B. Remember A, B should not overlap on each other. After joining A, B attach them with cello tape Fig(c).

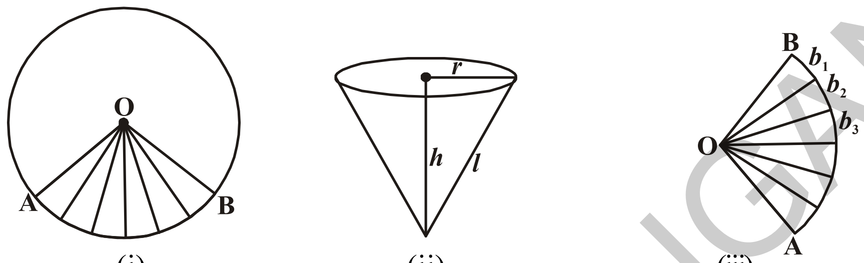


(iv) What kind of shape you have obtained?

Is it a right cone?

While making a cone observe what happened to the edges 'OA' and 'OB' and length of arc AB of the sector?

10.4.2 Curved Surface area of a cone



(i) (ii) (iii)

Let us find the surface area of a right circular cone that we made out of the paper as discussed in the activity.

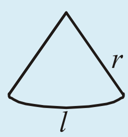
While folding the sector into cone you have noticed that OA, OB of sector coincides and becomes the slant height of the cone, whereas the length of $\overset{\frown}{AB}$ becomes the circumference of the base of the cone.

Now unfold the cone and cut the sector AOB as shown in the figure as many as you can, then you can see each cut portion is almost a small triangle with base b_1, b_2, b_3 ..... etc. and height 'l' i.e. equal to the slant height of the cone.

If we find the area of these triangles and adding these, it gives area of the sector. We know that sector forms a cone, so the area of a sector is equal to curved the surface area of the cone formed with it.

TRY THIS

A sector with radius 'r' and length of its arc 'l' is cut from a circular sheet of paper. Fold it as a cone. How can you derive the formula of its curved surface area $A = \pi rl$



Area of the cone = Sum of the areas of triangles.

$$= \frac{1}{2}b_1 l + \frac{1}{2}b_2 l + \frac{1}{2}b_3 l + \frac{1}{2}b_4 l + ...$$

$$= \frac{1}{2}l(b_1 + b_2 + b_3 + b_4 + ...)$$

$$= \frac{1}{2}l \text{ (length of the curved part from A to B or circumference of the base of the cone)}$$

$$= \frac{1}{2}l(2\pi r) \quad (\because b_1 + b_2 + b_3 + ... = 2\pi r, \text{ where 'r' is the radius of the cone})$$

as $\overset{\frown}{AB}$ forms a circle.

Thus, lateral surface area or curved surface area of the cone = πrl

Where 'l' is the slant height of the cone and 'r' is its radius of base.

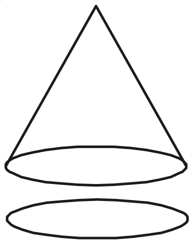
10.4.3 Total surface area of the cone

If the base of the cone is to be covered, we need a circle whose radius is equal to the radius of the cone.

How to obtain the total surface area of cone? How many surfaces you have to add to get total surface area?

The area of the circle $= \pi r^2$

Total surface area of a cone = lateral surface area + area of its base



$$= \pi r l + \pi r^2$$

$$= \pi r (l + r)$$

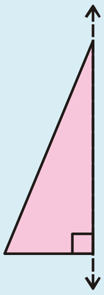
Total surface area of the cone = $\pi r (l + r)$

Where 'r' is the radius of the cone and 'l' is its slant height.

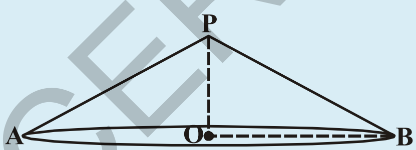
DO THIS

1. Cut a right angled triangle, stick a string along its perpendicular side, as shown in fig. (i) hold the both the sides of a string with your hands and rotate it with constant speed.

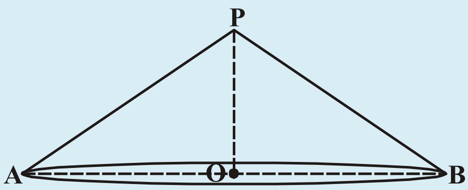
   What do you observe ?



2. Find the curved surface area and total surface area of each of the following Right Circular Cones.

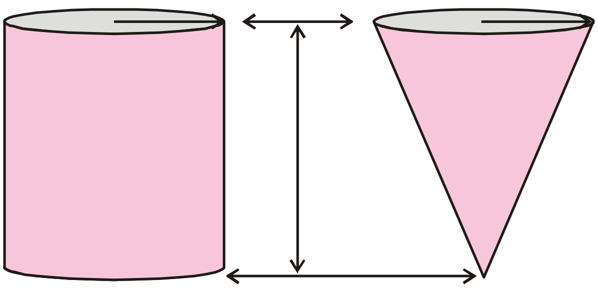


OP = 2cm.; OB = 3.5cm.

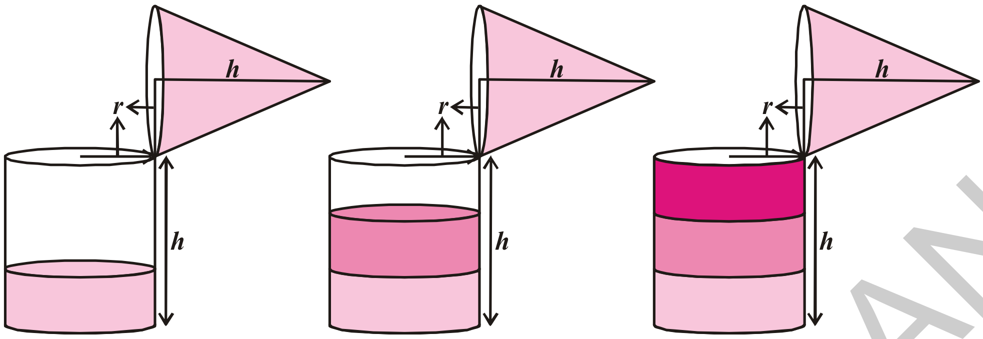


OP = 3.5cm.; AB = 10cm.

10.4.4 Volume of a right circular cone



Make a hollow cylinder and a hollow cone with the equal radius and equal height and do the following experiment, that will help us to find the volume of a cone.



i. Fill water in the cone up to the brim and pour into the hollow cylinder, it will fill up only some part of the cylinder.

ii. Again fill up the cone up to the brim and pour into the cylinder, we see the cylinder is still not full.

iii. When the cone is filled up for the third time and emptied into the cylinder, observe whether the cylinder is filled completely or not.

With the above experiment do you find any relation between the volume of the cone and the volume of the cylinder?

We can say that three times the volume of a cone makes up the volume of cylinder. Means volume of a cylinder is three times the volume of a cone if both have same radius and height.

So the volume of a cone is one third of the volume of the cylinder.

$$\therefore \text{ Volume of a cone} = \frac{1}{3}\pi r^2 h$$

where 'r' is the radius of the base of cone and 'h' is its height.

Example-6. A corn cob (see fig), shaped like a cone, has the radius of its broadest end as 1.4 cm and length (height) as 12 cm. If each 1cm^2 of the surface of the cob carries an average of four grains, find how many grains approximately you would find on the entire cob.

Solution : Here $l = \sqrt{r^2+h^2} = \sqrt{(1.4)^2+(12)^2}\,cm.$

$$= \sqrt{145.96} = 12.08 \text{ cm. (approx.)}$$

Therefore the curved surface area of the corn cob = πrl

$$= \frac{22}{7}\times 1.4\times 12.08\ cm^2$$

$$= 53.15 \text{ cm}^2$$
$$= 53.2 \text{ cm}^2 \text{ (approx)}$$

Number of grains of corn on 1 cm^2 of the surface of the corn cob = 4.

Therefore, number of grains on the entire curved surface of the cob.

$$= 53.2 \times 4 = 212.8 = 213 \text{ (approx)}$$

So, there would be approximately 213 grain of corn on the cob.

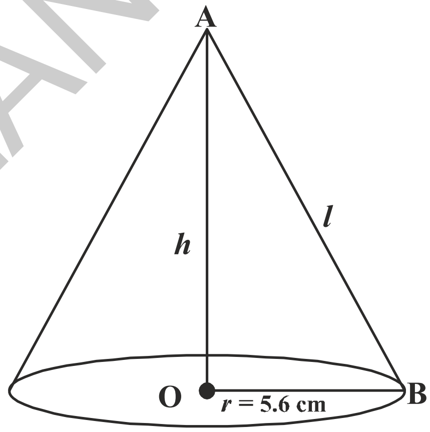
Example-7. Find the slant height and vertical height of a Cone with radius 5.6 cm and curved surface area 158.4 cm^2.

Solution : Radius = 5.6 cm, vertical height = h, slant height = l

CSA of cone $= \pi r l = 158.4 \text{ cm}^2$

$$\Rightarrow \frac{22}{7} \times 5.6 \times l = 158.4$$

$$\Rightarrow l = \frac{158.4 \times 7}{22 \times 5.6} = \frac{18}{2} = 9\text{cm}$$



we know $l^2 = r^2 + h^2$

$$h^2 = l^2 - r^2 = 9^2 - (5.6)^2$$
$$= 81 - 31.36$$
$$= 49.64$$
$$h = \sqrt{49.64}$$
$$h = 7.05 \text{ cm (approx)}$$

Example-8. A tent is in the form of a cylinder surmounted by a cone having its diameter of the base equal to 24 m. The height of cylinder is 11 m and the vertex of the cone is 5m above the cylinder. Find the cost of making the tent, if the rate of canvas is ₹10 per m^2.

Solution : Diametre of base of cylinder = diametre of cone = 24m

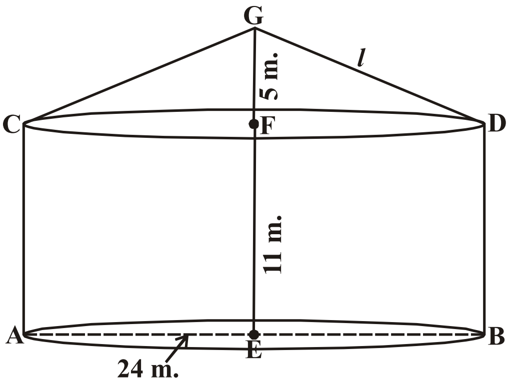
$\therefore$ Radius of base = 12 m

Height of cylinder = 11 m = h_1

Height of Cone = 5m = h_2

Let slant height of cone be l

$l = GD = \sqrt{r^2 + h^2} = \sqrt{12^2 + 5^2} = 13m$



Area of canvas required $= \text{CSA of cylinder} + \text{CSA of cone}$

$= 2\pi rh_1 + \pi rl$

$= \pi r\,(2h_1 + l)$

$= \frac{22}{7} \times 12(2 \times 11 + 13)m^2$

$= \frac{22 \times 12}{7} \times 35m^2$

$= 22 \times 60\ m^2$

$= 1320\ m^2$

Rate of canvas $=$ ₹10 per m^2

$\therefore$ Cost of canvas $=$ Rate $\times$ area of canvas

$=$ ₹10 $\times$ 1320

$=$ ₹13,200.

Example-9. A conical tent was erected by army at a base camp with height 3m. and base diameter 8m. Find;

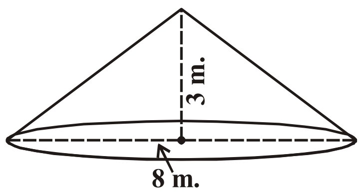
(i) The cost of canvas required for making the tent, if the canvas cost ₹ 70 per 1 sq.m.

(ii) If every person requires 3.5 $m.^3$ air, how many can be seated in that tent.

Solution : Diameter of the tent $= 8$ m.

$r = \frac{d}{2} = \frac{8}{2} = 4$ m.

height $= 3$ m.



Slant height $(l) = \sqrt{h^2 + r^2}$

$= \sqrt{3^2 + 4^2}$

$= \sqrt{25} = 5$ m.

$\therefore$ Curved surface area of tent$= \pi rl$

$= \frac{22}{7} \times 4 \times 5 = \frac{440}{7}\ m^2$

Volume of the cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \frac{22}{7} \times 4 \times 4 \times 3$

$= \frac{352}{7}$ m^3

(i) Cost of canvas required for the tent

= Curved surface area of tent × Unit cost

$= \frac{440}{7} \times 70$

= ₹4400

(ii) Number of persons can be seated in the tent

$= \frac{\text{Volume of a conical tent}}{\text{Amount of air required for each person}}$

$= \frac{352}{7} \div 3.5$

$= \frac{352}{7} \times \frac{1}{3.5} = 14.36$

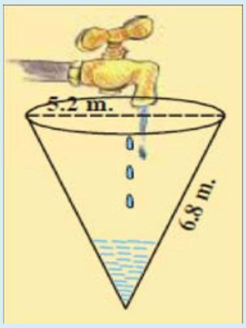
= 14 men (approx.)

Exercise - 10.3

1. The base area of a cone is 38.5 cm^2. Its volume is 77 cm^3. Find its height.
2. The volume of a cone is 462 m^3. Its base radius is 7 m. Find its height.
3. Curved surface area of a cone is 308 cm^2 and its slant height is 14 cm Find.

   (i) radius of the base (ii) Total surface area of the cone.
4. The cost of painting the total surface area of a cone at 25 paise per cm^2 is ₹176. Find the volume of the cone, if its slant height is 25 cm.
5. From a circle of radius 15 cm., a sector with angle 216° is cut out and its bounding radii are bent so as to form a cone. Find its volume.
6. The height of a tent is 9 m. Its base diameter is 24 m. What is its slant height? Find the cost of canvas cloth required if it costs ₹14 per sq.m.

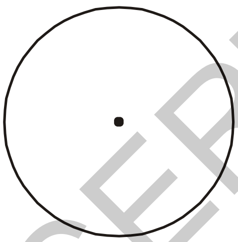
7. The curved surface area of a cone is $1159\frac{5}{7}$ cm^2. Area of its base is $254\frac{4}{7}$ cm^2. Find its volume.
8. A tent is cylindrical to a height of 4.8 m. and conical above it. The radius of the base is 4.5m. and total height of the tent is 10.8 m. Find the canvas required for the tent in square meters.
9. What length of tarpaulin 3 m wide will be required to make a conical tent of height 8m and base radius 6m? Assume that extra length of material that will be required for stitching margins and wastage in cutting is approximately 20 cm (use $\pi = 3.14$)
10. A Joker's cap is in the form of a right circular cone of base radius 7 cm and height 27 cm. Find the area of the sheet required to make 10 such caps.
11. Water is pouring into a conical vessel of diameter 5.2m and slant height 6.8m as shown in the adjoining figure, at the rate of 1.8 m^3 per minute. How long will it take to fill the vessel?



12. Two similiar cones have volumes 12π cu. units and 96π cu. units. If the curved surface area of the smaller cone is 15π sq. units, what is the curved surface area of the larger one?

Hint : For similar cones $\frac{r_1}{r_2} = \frac{h_1}{h_2} = \frac{l_1}{l_2} = \left(\frac{A_1}{A_2}\right)^3 = \left(\frac{V_1}{V_2}\right)^2$

10.5 Sphere


(i)


(ii)


(iii)

All the above figures are well known to you. Can you identify the difference among them?

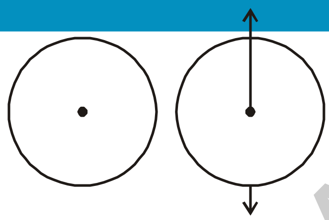
Figure (i) is a circle. You can easily draw it on a plane paper because it is a plane figure. A circle is plane closed figure whose every point lies at a constant distance (radius) from a fixed point (centre)

The remaining figures are solids. These solids are circular in shape and are called spheres.

A sphere is a three dimensional figure, which is made up of all points in the space, which is at a constant distance from a fixed point. This fixed point is called centre of the sphere. The distance from the centre to any point on the surface of the sphere is its radius.

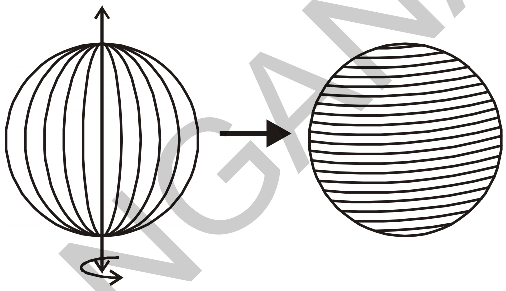
ACTIVITY

Draw a circle on a thick paper and cut it neatly. Stick a string along its diameter. Hold both the ends of the string with hands and rotate with constant speed and observe the figure so formed.

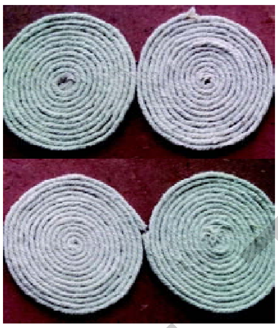


10.5.1 Surface area of a sphere

Let us find the surface area of the figure with the following activity.



Take a tennis ball as shown in the figure and wind a string around the ball, use pins to keep the string in place. Mark the starting and ending points of the string. Slowly remove the string from the surface of the sphere.



Find the radius of the sphere and draw four circles of radius equal to the radius of the ball as shown in the pictures. Start filling the circles one after one with the string you had wound around the ball.

What do you observe?

The string, which had completely covered the surface area of the sphere (ball), has been used to completely fill the area of four circles, all have same radius as of the sphere.

With this we can understand that the surface area of a sphere of radius (r) is equal to the four times of the area of a circle of radius (r).

Surface area of a sphere = 4 × the area of circle

$= 4\,\pi r^2$

$\therefore$ **Surface area of a sphere $= 4\,\pi r^2$**

Where 'r' is the radius of the sphere

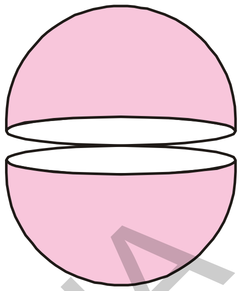
TRY THIS

Can you find the surface area of sphere in any other way?

10.5.2 Hemisphere

Take a solid sphere and cut it through the middle with a plane that passes through its centre.

Then it gets divided into two equal parts as shown in the figure



Each equal part is called a hemisphere.

A sphere has only one curved face. If it is divided into two equal parts, then its curved face is also divided into two equal curved faces.

What do you think about the surface area of a hemisphere ?

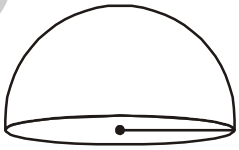
Obviously,

Curved surface area of a hemisphere is equal to half the surface area of the sphere

So, surface area of a hemisphere $= \frac{1}{2}$ surface area of sphere

$$= \frac{1}{2} \times 4\pi r^2$$

$$= 2\ \pi r^2$$



$\therefore$ **surface area of a hemisphere $= 2\ \pi r^2$**

The base of hemisphere is a circular region.

Its area is equal to $= \pi r^2$

Let us add both the curved surface area and area of the base, we get total surface area of hemisphere.

Total surface area of hemisphere $=$ Its curved surface area + area of its base

$$= 2\ \pi r^2 + \pi r^2$$

$$= 3\ \pi r^2.$$

$\therefore$ **Total surface area of hemisphere $= 3\pi r^2$.**

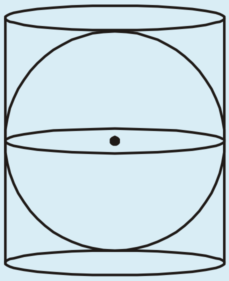
DO THESE

1. A right circular cylinder just encloses a sphere of radius r (see figure).

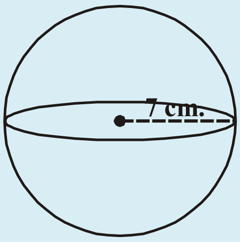
   Find : (i) surface area of the sphere

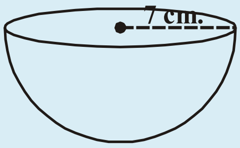
   (ii) curved surface area of the cylinder

   (iii) ratio of the areas obtained in (i) and (ii)



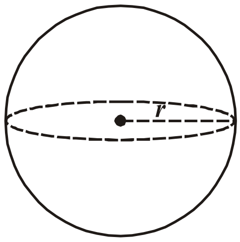
2. Find the surface area of the following figures.

(i) 

(ii) 

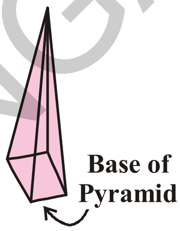
10.5.3 Volume of Sphere

To find the volume of a sphere, imagine that a sphere is composed of a great number of congruent pyramids with all their vertices join at the centre of the sphere, as shown in the figure.



(i)

(ii)



(iii)

Let us follow the steps:

1. Let ‘r’ be the radius of the solid sphere as in fig. (i).
2. Assume that a sphere with radius ‘r’ is made of ‘n’ number of pyramids of equal sizes as shown in the fig. (ii).
3. Consider a part (pyramid) among them. Each pyramid has a base and let the area of the base of pyramids are A_1, A_2, A_3.....

   The height of the pyramid is equal to the radius of sphere, then the

   Volume of one pyramid $= \frac{1}{3} \times \text{Area of the base} \times \text{height}$

   $= \frac{1}{3} A_1 r$

you can take any polygon as base of a pyramid

4. As there are ‘n’ number of pyramids, then

   Volume of ‘n’ pyramids $= \frac{1}{3} A_1 r + \frac{1}{3} A_2 r + \frac{1}{3} A_3 r + \ldots\ldots \text{ n times}$

   $= \frac{1}{3} r\,[A_1 + A_2 + A_3 + \ldots\ldots \text{ n times}]$

   $= \frac{1}{3} \times A\,r$

$A = A_1 + A_2 + A_3 + \ldots\ldots$ n times
= Surface areas of ‘n’ pyramids

5. As the sum of volumes of all these pyramids is equal to the volume of sphere and the sum of the areas of all the bases of the pyramids is very close to the surface area of the sphere, (i.e. $4\pi r^2$).

So, volume of sphere $= \frac{1}{3}(4\pi r^2)\, r$

$= \frac{4}{3}\pi r^3$ cub. units

$\therefore$ **Volume of a sphere** $= \frac{4}{3}\pi r^3$

Where 'r' is the radius of the sphere

How can you find volume of hemisphere? It is half the volume of sphere.

$\therefore$ **Volume of hemisphere** $= \frac{1}{2} \times$ volume of a sphere

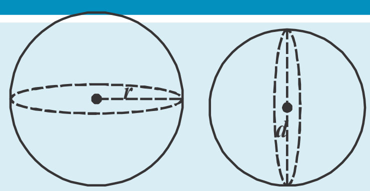
$$= \frac{1}{2} \times \frac{4}{3}\pi r^3$$

$$= \frac{2}{3}\pi r^3$$

[**Hint :** You can try to derive these formulae using water melon or any other like that]

Do This

1. Find the volume of the sphere given in the adjacent figures.
2. Find the volume of sphere of radius 6.3 cm.



$r = 3$cm. $d = 5.4$cm.

Example-10. If the surface area of a sphere is 154 cm^2, find its radius.

Solution : Surface area of sphere $= 4\pi r^2$

$$4\pi r^2 = 154 \Rightarrow 4 \times \frac{22}{7} \times r^2 = 154$$

$$\Rightarrow r^2 = \frac{154 \times 7}{4 \times 22} = \frac{7^2}{2^2}$$

$$\Rightarrow r = \frac{7}{2} = 3.5\,\text{cm}$$

Example-11. A hemispherical bowl is made up of stone whose thickness is 5 cm. If the inner radius is 35 cm, find the total surface area of the bowl.

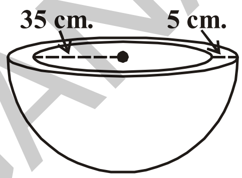
Solution : Let R be outer radius and 'r' be inner radius Thickness of ring = 5 cm

$\therefore$ R = (r + 5) cm = (35 + 5) cm = 40 cm

Total Surface Area = curved surface area of outer hemisphere + curved surface area of inner hemisphere + area of the ring.

$$= 2\pi R^2 + 2\pi r^2 + \pi(R^2 - r^2)$$

$$= \pi(2R^2 + 2r^2 + R^2 - r^2)$$



$$= \frac{22}{7}(3R^2 + r^2) = \frac{22}{7}(3 \times 40^2 + 35^2)\,\text{cm}^2$$

$$= \frac{6025 \times 22}{7}\,\text{cm}^2$$

$$= 18935.71\ \text{cm}^2\ \text{(approx)}.$$

Example-12. The hemispherical dome of a building needs to be painted (see fig 1). If the circumference of the base of dome is 17.6 m, find the cost of painting it, given the cost of painting is Rs.5 per 100 cm^2.

Solution : Since only the rounded surface of the dome is to be painted we need to find the curved surface area of the hemisphere to know the extent of painting that needs to be done. Now, circumference of base of the dome = 17.6 m $\therefore$ 17.6 = $2\pi r$

So, The radius of the dome $= 17.6 \times \frac{7}{2 \times 22}$ m

$= 2.8$ m

The curved surface area of the dome $= 2\pi r^2$

$$= 2 \times \frac{22}{7} \times 2.8 \times 2.8\,\text{m}^2$$

$$= 49.28\ \text{m}^2.$$

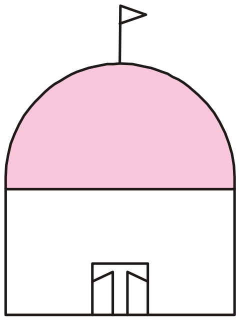


fig 1

Now, cost of painting 100 cm^2 is ₹ 5

So, cost of painting 1m^2 = ₹ 500

Therefore, cost of painting the whole dome

= ₹ 500 × 49.28

= ₹ 24640

Example-13. The hollow sphere, in which the circus motor cyclist performs his stunts, has a diameter of 7m. Find the area available to the motor cyclist for riding.

Solution : Diameter of the sphere = 7 m. Therefore, radius is 3.5 m. So, the riding space available for the motorcyclist is the surface area of the 'sphere' which is given by

$$4\pi r^2 = 4\times\frac{22}{7}\times 3.5\times 3.5 m^2$$

$$= 154 m^2.$$

Example-14. A shot put is a metallic sphere of radius 4.9 cm. If the density of the metal is 7.8 g. per cm^3, find the mass of the shot put.

Solution : Since the shot put is a solid sphere made of metal and its mass is equal to the product of its volume and density, we need to find the volume of the sphere.

Now, volume of the sphere $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3}\times\frac{22}{7}\times 4.9\times 4.9\times 4.9 \text{ cm}^3$$

$$= 493 \text{ cm}^3 \text{ (nearly)}$$

Further, mass of $1 cm^3$ of metal is 7.8 g

Therefore, mass of the shot put $= 7.8 \times 493$ g

$= 3845.44$g $= 3.85$ kg (nearly)

Example-15. A hemispherical bowl has a radius of 3.5 cm. What would be the volume of water it would contain ?

Solution : The volume of water the bowl can contains = Volume of hemisphere

$$= \frac{2}{3}\pi r^3$$

$$= \frac{2}{3}\times\frac{22}{7}\times 3.5\times 3.5\times 3.5 \text{ cm}^3$$

$$= 89.8 \text{ cm}^3. \text{ (approx)}.$$

Exercise - 10.4

1. The radius of a sphere is 3.5 cm. Find its surface area and volume.
2. The surface area of a sphere is $1018\frac{2}{7}$ sq.cm. What is its volume?
3. The length of equator of the globe is 44 cm. Find its surface area.
4. The diameter of a spherical ball is 21 cm. How much quantity of material is required to prepare 5 such balls?
5. The ratio of radii of two spheres is 2 : 3. Find the ratio of their surface areas and volumes.
6. Find the total surface area of a hemisphere of radius 10 cm. (use $\pi = 3.14$)
7. The diameter of a spherical balloon increases from 14 cm. to 28 cm. as air is being pumped into it. Find the ratio of surface areas of the balloons in the two cases.
8. A hemispherical bowl is made of brass, 0.25 cm. thickness. The inner radius of the bowl is 5 cm. Find the ratio of outer surface area to inner surface area.
9. The diameter of a lead ball is 2.1 cm. The density of the lead used is 11.34 g/c^3. What is the weight of the ball?
10. A metallic cylinder of diameter 5 cm. and height $3\frac{1}{3}$ cm. is melted and cast into a sphere. What is its diameter.
11. How many litres of milk can a hemispherical bowl of diameter 10.5 cm. hold?
12. A hemispherical bowl has diameter 9 cm. The liquid is poured into cylindrical bottles of diameter 3 cm. and height 3 cm. If a full bowl of liquid is filled in the bottles, find how many bottles are required.

What we have discussed?

1. Cuboid and cube are regular prisms having six faces and of which four are lateral faces and the base and top.
2. If length of cuboid is l, breadth is 'b' and height is 'h' then,

   Total surface area of a cuboid $= 2\,(lb + bh + lh)$

   Lateral surface area of a cuboid $= 2\,h\,(l + b)$

   Volume of a cuboid $= lbh$

F7S8F5

3. If the length of the edge of a cube is 'l' units, then

   Total surface area of a cube $= 6l^2$

   Lateral surface area of a cube $= 4l^2$

   Volume of a cube $= l^3$

4. The volume of a pyramid is $\frac{1}{3}$rd volume of a right prism if both have the same base and same height.

   Volume of pyramid $= \frac{1}{3} \times$ volume of right prism

5. A cylinder is a solid having two circular ends with a curved surface area. If the line segment joining the centres of base and top is perpendicular to the base, it is called right circular cylinder.

6. If the radius of right circular cylinder is 'r' and height is 'h' then;
   - Curved surface area of a cylinder $= 2\pi rh$
   - Total surface area of a cylinder $= 2\pi r (r + h)$
   - Volume of a cylinder $= \pi r^2 h$

7. Cone is a geometrical shaped object with circle as base, having a vertext at the top. If the line segment joining the vertex to the centre of the base is perpendicular to the base, it is called right circular cone.

8. The length joining the vertex to any point on the circular base of the cone is called slant height (l)

   $l^2 = h^2 + r^2$

9. If 'r' is the radius, 'h' is the height, 'l' is the slant height of a cone, then
   - Curved surface area of a cone $= \pi r l$
   - Total surface area of a cone $= \pi r (r + l)$

10. The volume of a cone is $\frac{1}{3}$rd the volume of a cylinder of the same base and same height

    i.e. volume of a cone $= \frac{1}{3} \pi r^2 h$.

11. A sphere is an geometrical object formed where the set of points are equidistant from the fixed point in the space. The fixed point is called centre of the sphere and the fixed distance is called radius of the sphere.

12. If the radius of sphere is 'r' then,

- Surface area of a sphere = $4\pi r^2$
- Volume of a sphere = $\frac{4}{3}\pi r^3$

13. A plane through the centre of a sphere divides it into two equal parts, each of which is called a hemisphere.

- Curved surface area of a hemisphere = $2\pi r^2$
- Total surface area of a hemisphere = $3\pi r^2$
- Volume of a hemisphere = $\frac{2}{3}\pi r^3$

Do You Know?

Making an 8 × 8 Magic Square

Simply place the numbers from 1 to 64 sequentially in the square grids, as illustrated in fig.-(i). Sketch in the dashed diagonals as indicated. To obtain the magic square as in fig.-(ii), replace any number which lands on a dashed line with its compliment (two numbers of a magic square are compliments if they total the same value as the sum of the magic's square smallest and largest numbers).

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

fig.-(i)

64	2	3	61	60	6	7	57
9	55	54	12	13	51	50	16
17	47	46	20	21	43	42	24
40	26	27	37	36	30	31	33
32	34	35	29	28	38	39	25
41	23	22	44	45	19	18	48
49	15	14	52	53	11	10	56
8	58	59	5	4	62	63	1

fig.-(ii)

* A magic square is an array of numbers arrange in a square shape in which any row, column total the same amount. You can try more such magic squares.

Chapter

11

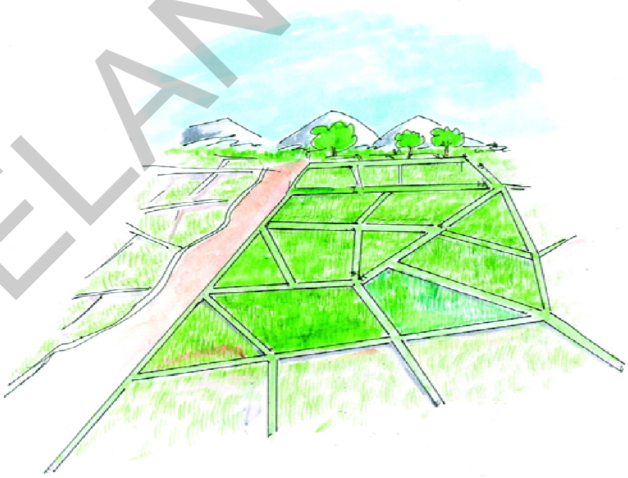
Areas

B4Y5K6

11.1 Introduction

Have you seen agricultural fields around your village or town? The land is divided amongst various farmers and there are many fields. Are all the fields of the same shape and same size? Do they have the same area? If a field has to be further divided among some persons, how will they divide it? If they want equal area, what can they do?

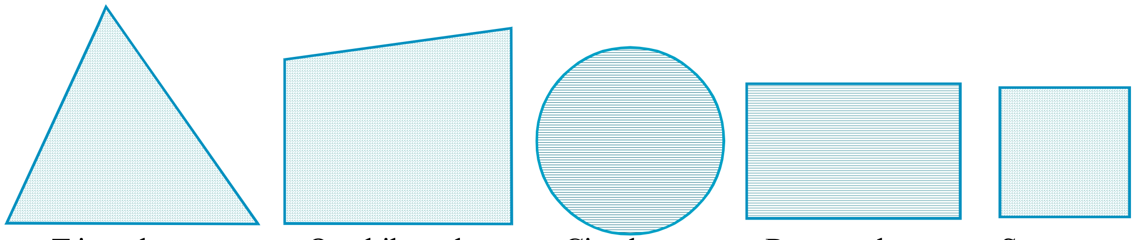
How does a farmer estimate the amount of fertilizer or seed needed for field? Does the size of the field have anything to do with this number?



The earliest and the most important reason for the initiation of the study of geometry is agricultural organisation. This includes measuring the land, dividing it into appropriate parts and recasting boundaries of the fields for the sake of convenience. In history you may have discussed the floods of river Nile (Egypt) and the land markings generated later . Some of these fields resemble the basic shapes such as square, rectangle trapezium, parallelograms etc., and some are in irregular shapes. For the basic shapes, we follow the rules to find areas from given measurements. We would study some of them in this chapter. We will learn how to calculate areas of triangles, squares, rectangles and quadrilaterals by using formulae. We will also explore the basis of those formulae. We will discuss how are they derived ? What do we mean by 'area'?

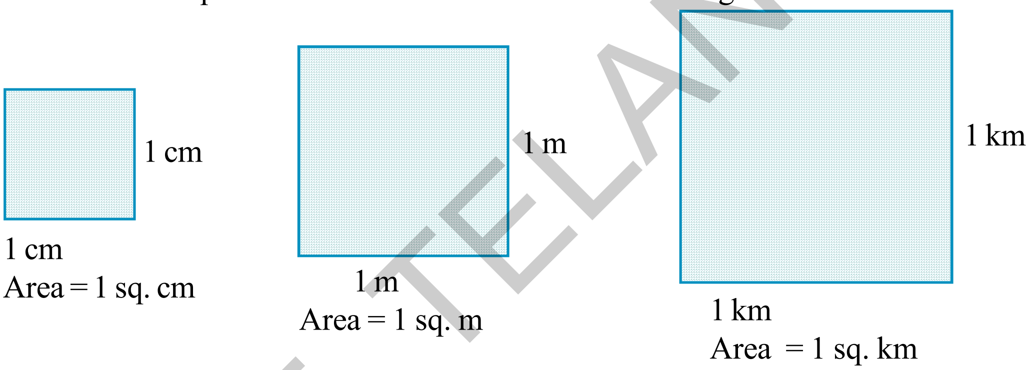
11.2 Area of Planar Regions

You may recall that the part of the plane enclosed by a simple closed figure is called a planar region corresponding to that figure. The magnitude or measure of this planar region is its area.



A planar region consists of a boundary and an interior region. How can we measure the area of this? The magnitude of measure of these regions (i.e. areas) is always expressed with a positive real number (in some unit of area) such as 10 cm^2, 215 m^2, 2 km^2, 3 hectares etc. So, we can say that area of a figure is a number (in some unit of area) associated with the part of the plane enclosed by the figure.

The unit area is the area of a square of a side of unit length. Hence **square centimeter** (or 1cm^2) is the area of a square drawn on a side one centimeter in length.

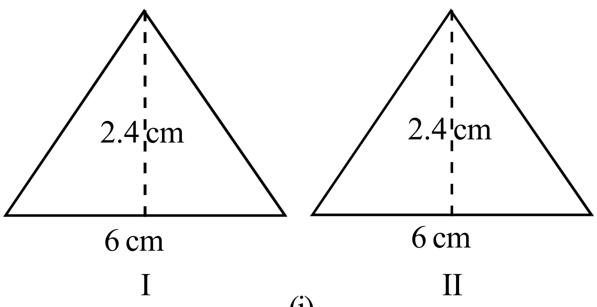


The terms square meter (1m^2), square kilometer (1km^2), square millimeter (1mm^2) are to be understood in the same sense. We are familiar with the concept of congruent figures from earlier classes. Two figures are congruent if they have the same shape and the same size.

Activity

Observe Figure I and II. Find the area of both figures. Are the areas equal?

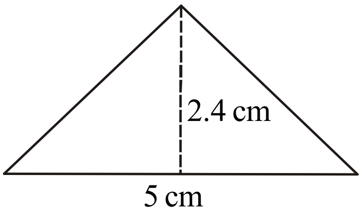
Trace these figures on a sheet of paper, cut them. Cover fig. I with fig. II. Do they cover each other completely? Are they congruent?



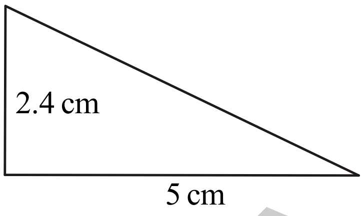
(i)

Observe fig. III and IV. Find the areas of both. What do you notice?

Are they congruent?

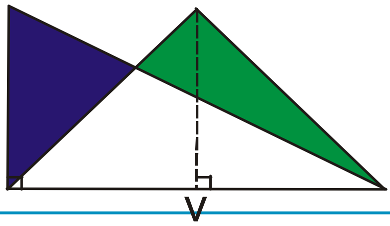


III



IV

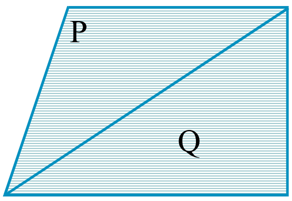
Now, trace these figures on sheet of paper. Cut them let us cover fig. III by fig. IV by conciding their bases (length of same side).


V

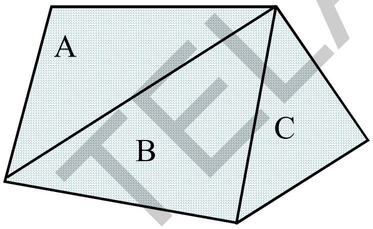
As shown in figure V are they covered completely?

We conclude that Figures I and II are congruent and equal in area. But figures III and IV are equal in area but they are not congruent.

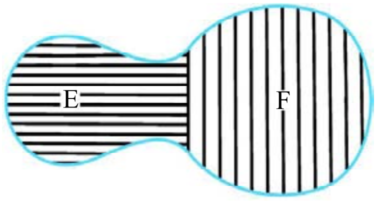
Now consider the figures given below:



X



Y



Z

You may observe that planar region of figures X, Y, Z is made up of two or more planar regions. We can easily see that

Area of figure X = Area of figure P + Area of figure Q.

Similarly area of (Y) = area of (A) + area of (B) + area of (C)

area of (Z) = area of (E) + area of (F).

Thus the area of a figure is a number (in some units) associated with the part of the plane enclosed by the figure with the following properties.

(**Note :** We use area of a figure (X) briefly as ar(X) from now onwards)

(i) The areas of two congruent figures are equal.

If A and B are two congruent figures, then ar(A) = ar(B)

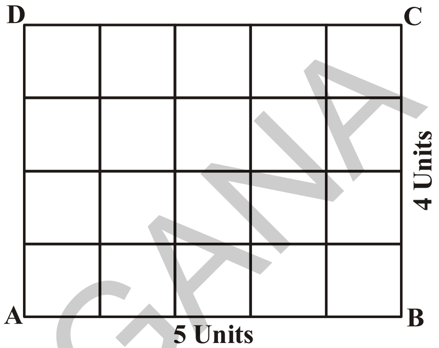
(ii) The area of a figure is equal to the sum of the areas of finite number of parts of it.
If a planar region formed by a figure X is made up of two non-overlapping planar regions formed by figures P and Q then ar(X) = ar(P) + ar(Q).

11.3 Area of Rectangle

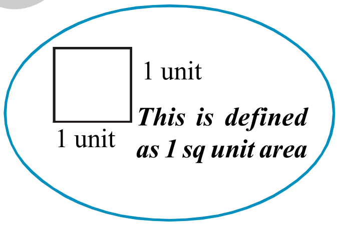
If the number of units in the length of a rectangle is multiplied by the number of units in its breadth, the product gives the number of square units in the area of rectangle

Let ABCD represent a rectangle whose length AB is 5 units and breadth BC is 4 units.

Divide AB into 5 equal parts and BC into 4 equal parts and through the points of division of each line draw parallels to the other. Each compartment in the rectangle represents one square unit (why ?)



$\therefore$ The rectangle contains 5 units $\times$ 4 units. That is 20 square units.



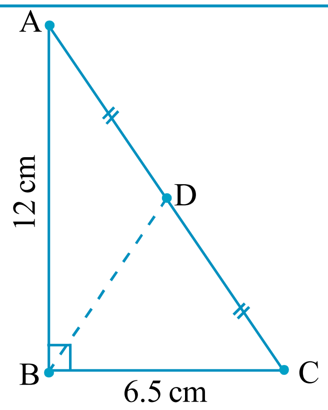
Similarly, if the length is '*l*' units and breadth is 'b' units then the area of rectangle is '*l*b' square units. That is "length $\times$ breadth" square units gives the area of a rectangle.

Think, Discuss and Write

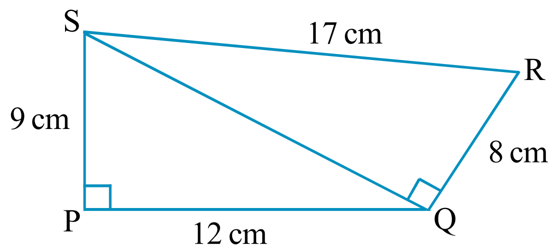
1. If 1cm represents 5m, what would be an area of 6 square cm. represent ?
2. Rajni says 1 sq.m = 100^2 sq.cm. Do you agree? Explain.

Exercise - 11.1

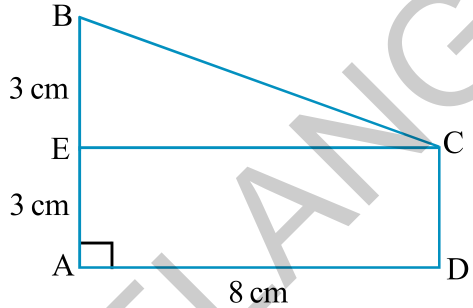
1. In ΔABC, $\angle ABC = 90°$, $AD = DC$, $AB = 12cm$ and $BC = 6.5$ cm. Find the area of ΔADB.



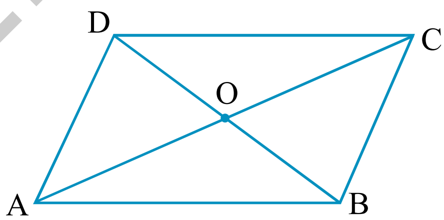
2. Find the area of a quadrilateral PQRS in which $\angle QPS = \angle SQR = 90°$, PQ = 12 cm, PS = 9 cm, QR = 8 cm and SR = 17 cm (**Hint:** PQRS has two parts)



3. Find the area of trapezium ABCD as given in the figure in which ADCE is a rectangle. (**Hint:** ABCD has two parts)



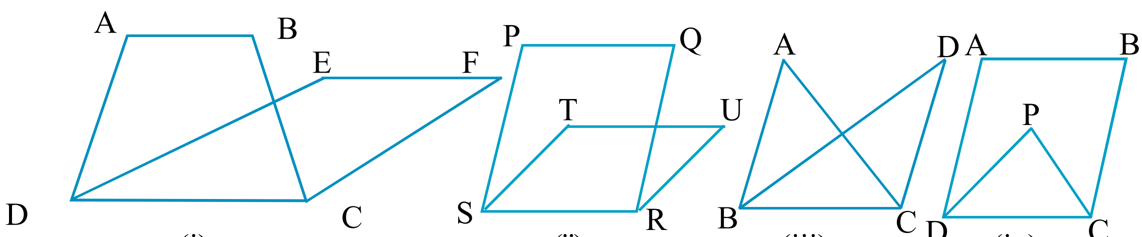
4. ABCD is a parallelogram. The diagonals AC and BD intersect each other at 'O'. Prove that ar(ΔAOD) = ar(ΔBOC). (**Hint:** Congruent figures have equal area)



11.4 Figures on the same base and between the same parallels

We shall now study some relationships between the areas of some geometrical figures under the condition that they lie on the same base and between the same parallels. This study will also be useful in understanding of some results on similarity of triangles.

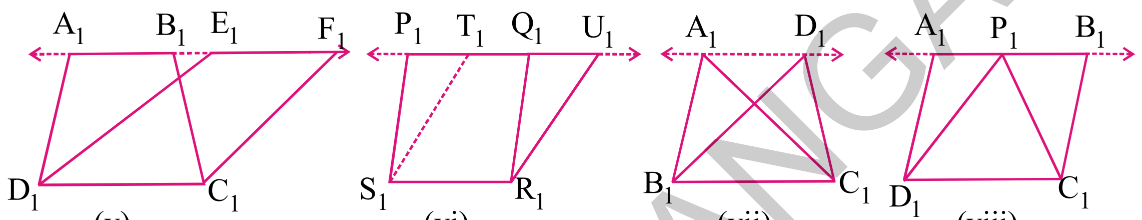
Look at the following figures.



(i) (ii) (iii) (iv)

In Fig(i) a trapezium ABCD and parallelogram EFCD have a common side CD. We say that trapezium ABCD and parallelogram EFCD are on the same base CD. Similarly in fig(ii) the base of parallelogram PQRS and parallelogram TURS is the same. In fig(iii) Triangles ABC and DBC have the same base BC. In Fig(iv) parallelogram ABCD and triangle PCD lie on DC so, all these figures are of geometrical shapes are therefore on the same base. They are however not between the same parallels as AB does not overlap EF and PQ does not overlap TU etc. Neither the points A, B, E, F are collinear nor the points P, Q, T, U. What can you say about Fig(iii) and Fig (iv)?

Now observe the following figures.



(v) (vi) (vii) (viii)

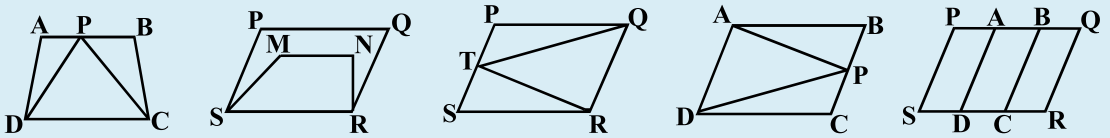
What difference have you observed among the figures? In Fig(v), We say that trapezium $A_1B_1C_1D_1$ and parallelogram $E_1F_1C_1D_1$ are on the same base and between the same parallels A_1F_1 and D_1C_1. The points A_1, B_1, E_1, F_1 are collinear and $A_1F_1 \parallel D_1C_1$. Similarly in fig. (vi) parallelograms $P_1Q_1R_1S_1$ and $T_1U_1R_1S_1$ are on the same base S_1R_1 and between the same parallels P_1U_1 and S_1R_1. Name the other figures on the same base and the parallels between which they lie in fig. (vii) and (viii).

So, two figures are said to be on the same base and between the same parallels, if they have a common base (side) and the vertices (or the vertex) opposite to the common base of each figure lie on a line parallel to the base.

Think, Discuss and Write

Which of the following figures lie on the same base and between the same parallels?

In such a cases, write the common base and the two parallels.



(a) (b) (c) (d) (e)

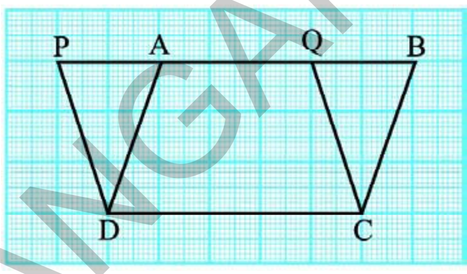
11.5 PARALLELOGRAMS ON THE SAME BASE AND BETWEEN THE SAME PARALLELS

Now let us try to find a relation, if any, between the areas of two parallelograms on the same base and between the same parallels. For this, let us perform the following activity.

ACTIVITY

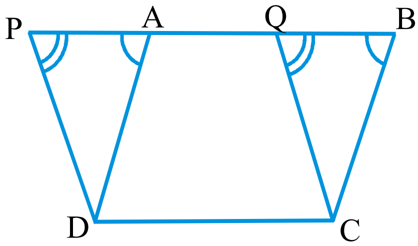
Take a graph sheet and draw two parallelograms ABCD and PQCD on it as show in the Figure-

The parallelograms are on the same base DC and between the same parallels PB and DC Clearly the part DCQA is common between the two parallelograms. So if we can show that ΔDAP and ΔCBQ have the same area then we can say ar(PQCD) = ar(ABCD).



Theorem-11.1 : Parallelograms on the same base and between the same parallels are equal in area.

Proof: Let ABCD and PQCD are two parallelograms on the samebase DC and between the parallel lines DC and PB.



In ΔDAP and ΔCBQ

PD || CQ and PB is transversal $\angle DPA = \angle CQB$

and AD || CB and PB is transversal $\angle DAP = \angle CBQ$

also PD = QC as PQCD is a parallelogram.

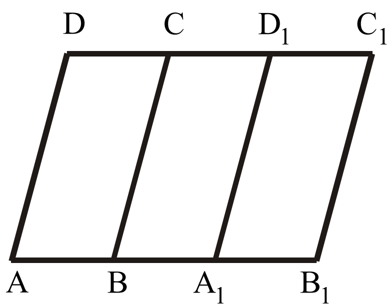
Hence ΔDAP and ΔCBQ are congruent and have equal areas.

So we can say ar(PQCD) = ar (AQCD) + ar(DAP)

= ar(AQCD) + ar(CBQ) = ar(ABCD)

You can verify by counting the squares of these parallelogram as drawn in the graph sheet.

Can you explain how to count full squares below half a square, above half a square on graph sheet.



Reshma argues that the parallelograms between same parallels need not have a common base for equal area. They only need to have an equal base. To understand her statement look at the adjacent figure.

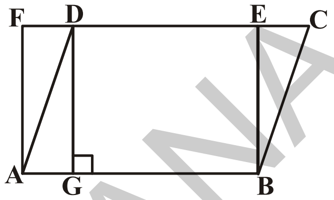
If AB = A_1B_1 When we cut out parallelogram $A_1B_1C_1D_1$ and place it over parallelogram ABCD, A would concide in with A_1 and B with B_1 and C_1, D_1 coincide with C, D. Thus these are equal in area. Thus the parallelogram with the equal base can be considered to be on the same base for the purposes of studying their geometrical properties.

Let us now take some examples to illustrate the use of the above Theorem.

Example-1. ABCD is parallelogram and ABEF is a rectangle and DG is perpendicular on AB.

Prove that (i) ar (ABCD) = ar(ABEF)

(ii) ar (ABCD) = AB × DG



Solution : (i) A rectangle is also a parallelogram

$\therefore$ ar(ABCD) = ar(ABEF) ..... (1)

(Parallelograms lie on the same base and between the same parallels)

(ii) ar(ABCD) = ar(ABEF) ($\because$ from (1))

= AB × BE ($\because$ ABEF is a rectangle)

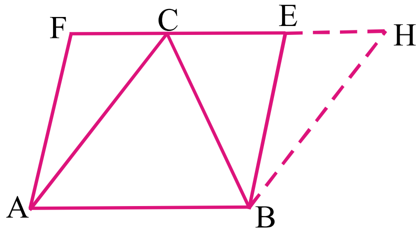
= AB × DG ($\because$ DG $\perp$ AB and DG = BE)

Therefore ar(ABCD) = AB × DG

From the result, we can say that "area of a parallelogram is the product of its any side and the corresponding altitude".

Example-2. Triangle ABC and parallelogram ABEF are on the same base, AB as in between the same parallels AB and EF. Prove that ar(ΔABC) = $\frac{1}{2}$ar(|| gm ABEF)

Solution : Through B draw BH || AC to meet FE produced at H



$\therefore$ ABHC is a parallelogram

Diagonal BC divides it into two congruent triangles

$\therefore$ ar(ΔABC) = ar(ΔBCH)

= $\frac{1}{2}$ar (|| gm ABHC)

But || gm ABHC and || gm ABEF are on the same base AB and between same parallels AB and EF

$\therefore$ ar(|| gm ABHC) = ar(|| gm ABEF)

Hence ar(ΔABC) = $\frac{1}{2}$ar (|| gm ABEF)

From the result, we say that "the area of a triangle is equal to half the area of the parallelogram on the same base and between the same parallels".

Example-3. Find the area of a figure formed by joining the mid-points of the adjacent sides of a rhombus with diagonals 12 cm. and 16 cm.

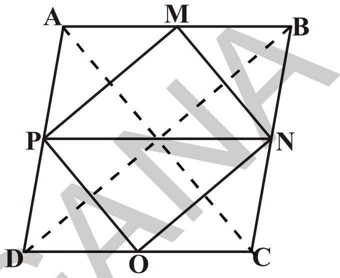
Solution : Join the mid points of AB, BC, CD, DA of a rhombus ABCD and name them M, N, O and P respectively to form a figure MNOP.

What is the shape of MNOP thus formed? Give reasons?

Join P, N, then PN || AB and PN || DC (How?)

We know that if a triangle and a parallelogram are on the same base and between the same parallels, the area of the triangle is equal to one-half area of the parallelogram.

From the above result parallelogram ABNP and triangle MNP are on the same base PN and in between same parallel lines PN and AB.



$\therefore$ ar $\Delta MNP = \frac{1}{2}$ ar ABPN .....(i)

Similarly ar $(\Delta PON) = \frac{1}{2}$ ar (PNCD).....(ii)

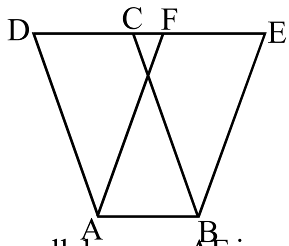
and Area of rhombus $= \frac{1}{2} \times d_1 d_2$ .....(iii)

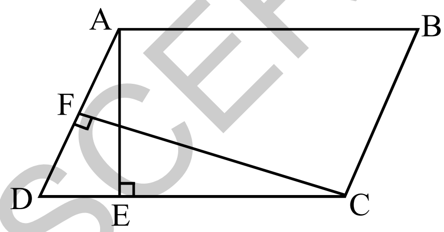
From (1), (ii) and (iii) we get

$$\begin{aligned} ar(MNOP) &= ar(\Delta MNP) + ar(\Delta PON) \\ &= \frac{1}{2} ar(ABNP) + \frac{1}{2} ar(PDCN) \\ &= \frac{1}{2} ar(\text{rhombus } ABCD) \\ &= \frac{1}{2}\left(\frac{1}{2} \times 12 \times 16\right) = 48 \text{ cm.}^2 \end{aligned}$$

Exercise - 11.2

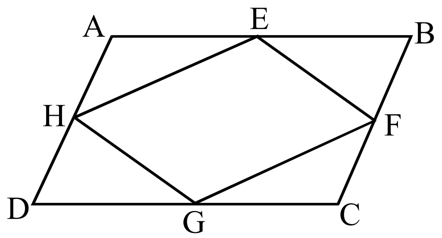
1. The area of parallelogram ABCD is 36 cm^2. Calculate the height of parallelogram ABEF if AB = 4.2 cm.





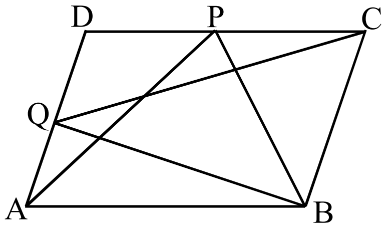
2. ABCD is a parallelogram. AE is perpendicular on DC and CF is perpendicular on AD.

   If AB = 10 cm, AE = 8 cm and CF = 12 cm. Find AD.



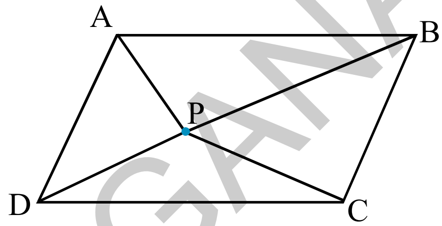
3. If E, F G and H are respectively the midpoints of the sides AB, BC, CD and AD of a parallelogram ABCD, show that ar(EFGH) $= \frac{1}{2}$ar(ABCD).

4. What type of quadrilateral do you get, if you join ΔAPM, ΔDPO, ΔOCN and ΔMNB in the example 3.



5. P and Q are any two points lying on the sides DC and AD respectively of a parallelogram ABCD show that ar(ΔAPB) = ar Δ(BQC).

6. P is a point in the interior of a parallelogram ABCD. Show that

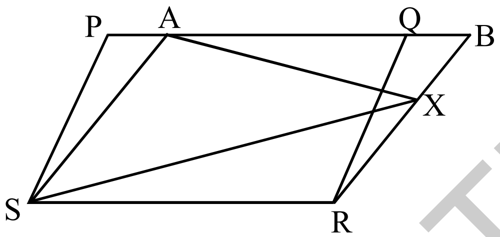


(i) $ar(\Delta APB) + ar(\Delta PCD) = \frac{1}{2} ar(ABCD)$

(ii) $ar(\Delta APD) + ar(\Delta PBC) = ar(\Delta APB) + ar(\Delta PCD)$

(Hint : Through P, draw a line parallel to AB)

7. Prove that the area of a trapezium is half the sum of the parallel sides multiplied by the distance between them.



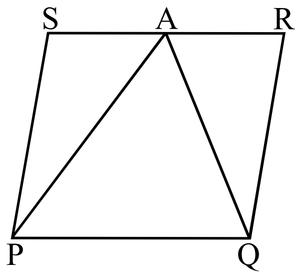
8. PQRS and ABRS are parallelograms and X is any point on the side BR. Show that

(i) ar(PQRS) = ar(ABRS)

(ii) $ar(\Delta AXS) = \frac{1}{2} ar(PQRS)$

9. A farmer has a field in the form of a parallelogram PQRS as shown in the figure. He took the mid- point A on RS and joined it to points P and Q. In how many parts of field is divided? What are the shapes of these parts ?

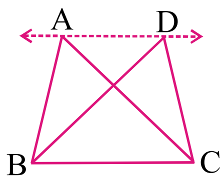
The farmer wants to sow groundnuts which are equal to the sum of pulses and paddy. How should he sow? State reasons?



10. Prove that the area of a rhombus is equal to half of the product of the diagonals.

11.6 Triangles on the same base and between the same parallels

We are looking at figures that lie on the same base and between the same parallels. Let us have two triangles ABC and DBC on the same base BC and between the same parallels, AD and BC.



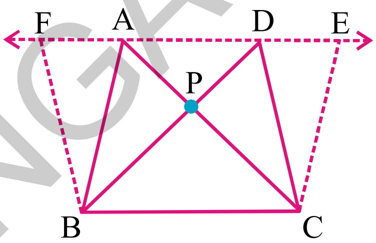
What can we say about the areas of such triangles? Clearly there can be infinite number of ways in which such pairs of triangle between the same parallels and on the same base can be drawn.

Let us perform an activity.

ACTIVITY

Draw pairs of triangles one the same base or (equal bases) and between the same parallels on the graph sheet as shown in the Figure.

Let ΔABC and ΔDBC be the two triangles lying on the same base BC and between parallels BC and AD. Extend AD on either sides and draw CE || AB and BF || CD. Parallelograms AECB and FDCB are on the same base BC and are between the same parallels BC and EF.



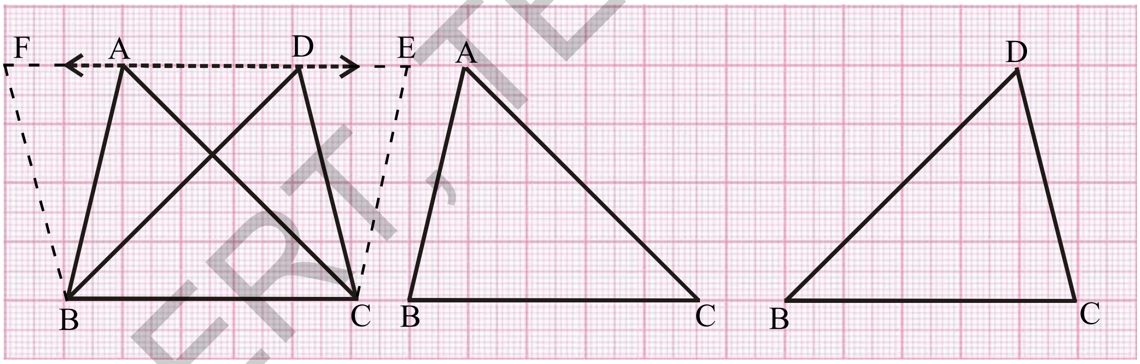
Thus ar(AECB) = ar(FDCB). (How ?)

We can see ar(ΔABC) = $\frac{1}{2}$ ar(Parallelogram AECB) ...(i)

and ar(ΔDBC) = $\frac{1}{2}$ ar(Parallelogram FDCB) .... (ii)

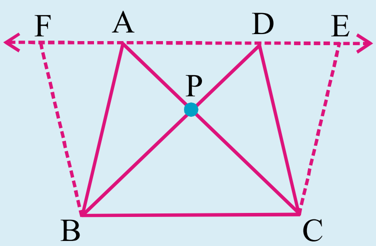
From (i) and (ii), we get ar(ΔABC) = ar(ΔDBC).

You can also find the areas of ΔABC and ΔDBC by the method of counting the squares in graph sheet as we have done in the earlier activity and check the areas are whether same.



THINK, DISCUSS AND WRITE

Draw two triangles ABC and DBC on the same base and between the same parallels as shown in the figure with P as the point of intersection of AC and BD. Draw CE || BA and BF || CD such that E and F lie on line AD.



Can you show ar(ΔPAB) = ar(ΔPDC)?

(**Hint** : These triangles are not congruent but have equal areas.)

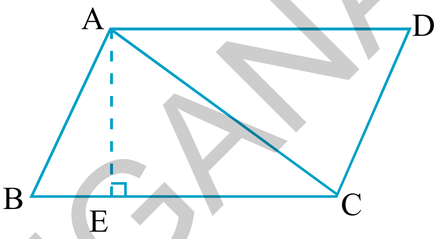
Corollary-1 : Show that the area of a triangle is half the product of its base (or any side) and the corresponding attitude (height).

Proof : Let ABC be a triangle. Draw AD || BC such that CD = BA.

Now ABCD is a parallelogram one of whose diagonals is AC.

We know $\Delta ABC \cong \Delta ACD$

So $ar(\Delta ABC) = ar(\Delta ACD)$ (Congruent triangles have equal area)



Therefore, $ar\Delta ABC = \frac{1}{2} ar(ABCD)$

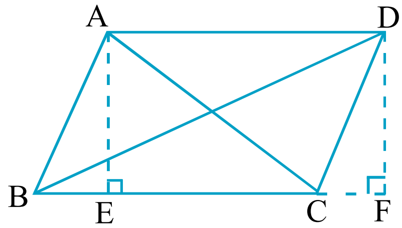
Draw $AE \perp BC$

We know $ar(ABCD) = BC \times AE$

We have $ar(\Delta ABC) = \frac{1}{2} ar(ABCD) = \frac{1}{2} \times BC \times AE$

So $ar\Delta ABC = \frac{1}{2} \times$ base BC $\times$ Corresponding attitude AE.

Theorem-11.2 : Two triangles having the same base (or equal bases) and equal areas will lie between the same parallels.



Observe the figure. Name the triangles lying on the same base BC. What are the heights of ΔABC and ΔDBC?

If two triangles have equal area and equal base, what will be their heights? Are A and D collinear?

Let us now take some examples to illustrate the use of the above results.

Example 4. Show that the median of a triangle divides it into two triangles of equal areas.

Solution : Let ABC be a triangle and Let AD be one of its medians.

In ΔABD and ΔADC the vertex is common and these bases BD and DC are equal.

Draw $AE \perp BC$.

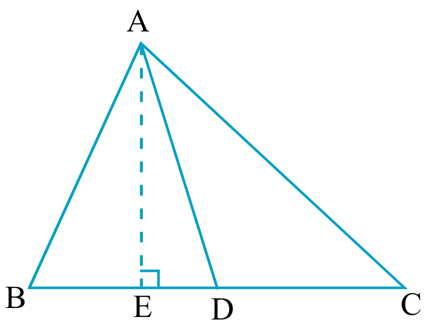
Now $ar(\Delta ABD) = \frac{1}{2} \times$ base BD $\times$ altitude of ΔADB

$= \frac{1}{2} \times BD \times AE$

$= \frac{1}{2} \times DC \times AE$ $(\because BD = DC)$

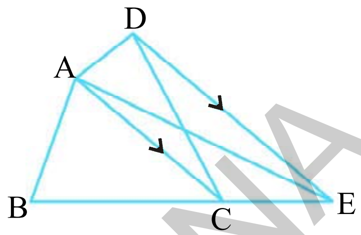
$= \frac{1}{2} \times$ base DC $\times$ altitude of ΔACD

$= ar\,\Delta ACD$



Hence $ar(\Delta ABD) = ar(\Delta ACD)$

Example-5. In the figure, ABCD is a quadrilateral. AC is the diagonal and DE || AC and also DE meets BC produced at E. Show that ar(ABCD) = ar(ΔABE).



Solution : ar(ABCD) = ar(ΔABC) + ar(ΔDAC)

ΔDAC and ΔEAC lie on the same base $\overline{AC}$

and between the parallels DE || AC

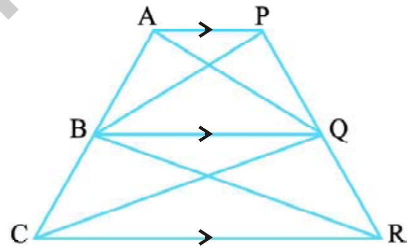
ar(ΔDAC) = ar(ΔEAC) (Why?)

Adding areas of same figures on both sides.

ar(ΔDAC) + ar(ΔABC) = ar(ΔEAC) + ar(ΔABC)

Hence ar(ABCD) = ar(ΔABE)

Example 6. In the figure, AP || BQ || CR. Prove that ar(ΔAQC) = ar(ΔPBR).



Solution : ΔABQ and Δ PBQ lie on the same base BQ and between the same parallels AP || BQ.

$\therefore$ ar(ΔABQ) = ar(ΔPBQ) ...(1)

Similarly,

ar (ΔCQB) = ar (ΔRQB) (same base BQ and BQ || CR) ...(2)

Adding results (1) and (2)

ar(ΔABQ) + ar(ΔCQB) = ar(ΔPBQ) + ar(ΔRQB)

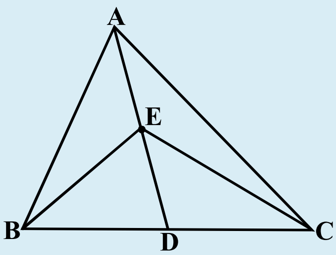
Hence ar(ΔAQC) = ar (ΔPBR)

EXERCISE - 11.3

1. In a triangle ABC (see figure), E is the midpoint of median AD, show that

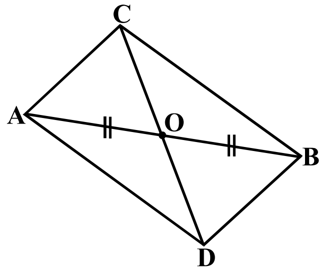
(i) ar ΔABE = ar ΔACE

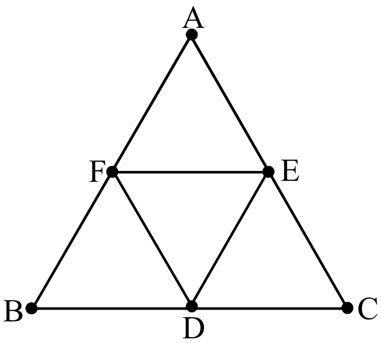
(ii) arΔABE $= \frac{1}{4}$ar(ΔABC)



2. Show that the diagonals of a parallelogram divide it into four triangles of equal area.

3. In the figure, ΔABC and ΔABD are two triangles on the same base AB. If line segment CD is bisected by $\overline{AB}$ at O, show that ar (ΔABC) = ar (ΔABD).





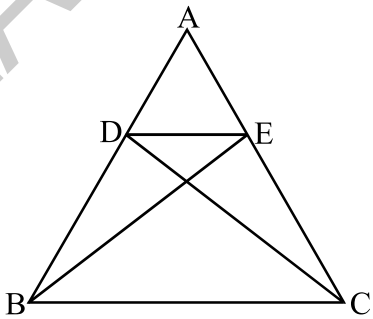
4. In the figure, ΔABC, D, E, F are the midpoints of sides BC, CA and AB respectively. Show that

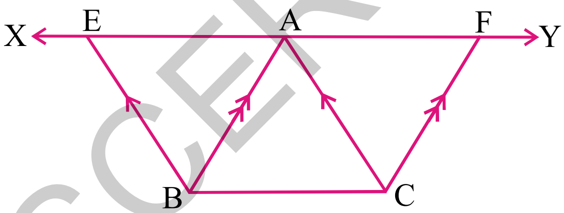
(i) BDEF is a parallelogram

(ii) $ar(\Delta DEF) = \frac{1}{4} ar(\Delta ABC)$

(iii) $ar(BDEF) = \frac{1}{2} ar(\Delta ABC)$

5. In the figure D, E are points on the sides AB and AC respectively of ΔABC such that $ar(\Delta DBC) = ar(\Delta EBC)$. Prove that DE || BC.

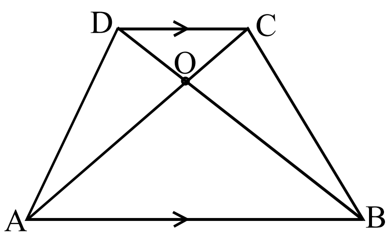


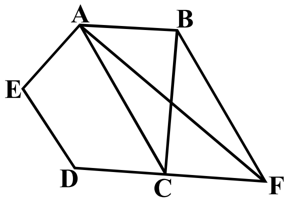


6. In the figure, XY is a line parallel to BC is drawn through A. If BE || CA and CF || BA are drawn to meet XY at E and F respectively. Show that $ar(\Delta ABE)$ = ar (ΔACF).

7. In the figure, diagonals AC and BD of a trapezium ABCD with AB || DC intersect each other at O. Prove that

$ar(\Delta AOD) = ar(\Delta BOC)$.

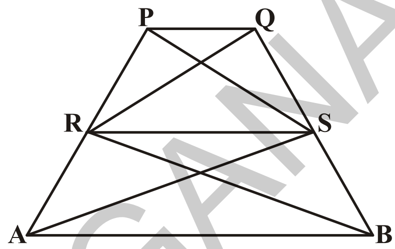




8. In the figure, ABCDE is a pentagon. A line through B parallel to AC meets DC produced at F. Show that

 (i) ar (ΔACB) = ar (ΔACF)

 (ii) ar (AEDF) = ar (ABCDE)

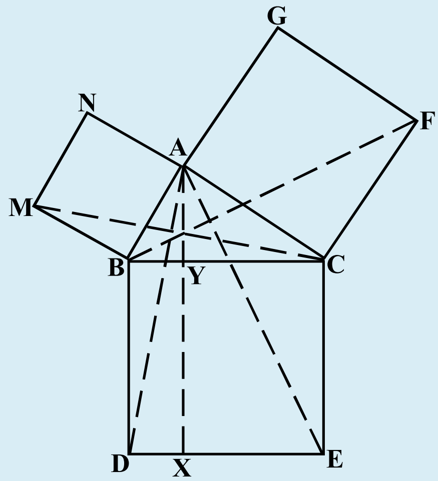


9. In the figure, if ar (ΔRAS) = ar (ΔRBS) and [ar (ΔQRB) = ar(ΔPAS) then show that both the quadrilaterals PQSR and RSBA are trapeziums.

10. A villager Ramayya has a plot of land in the shape of a quadrilateral. The grampanchayat of the village decided to take over some portion of his plot from one of the corners to construct a school. Ramayya agrees to the above proposal with the condition that he should be given equal amount of land in exchange of his land adjoining his plot so as to form a triangular plot. Explain how this proposal will be implemented. (Draw a rough sketch of plot).

Think, Discuss and Write

ABC is a right triangle right angled at A. BCED, ACFG and ABMN are squares on the sides BC, CA and AB respectively. Line segments AX $\perp$ DE meets BC at Y. and DE at X. Join AD, AE also BF and CM (See the figure).



Show that

(i) ΔMBC ≅ ΔABD

(ii) ar(BYXD) = 2ar (ΔMBC)

(iii) ar(BYXD) = ar(ABMN)

(iv) ΔFCB ≅ ΔACE

(v) ar(CYXE) = 2 ar(FCB)

(vi) ar(CYXE) = ar (ACFG)

(vii) ar(BCED) = ar(ABMN) + ar(ACFG)

Can you write the result (vii) in words ? This is a famous theorem of Pythagoras. You shall learn a simpler proof in this theorem in class X.

What have we discussed?

In this chapter we have discussed the following.

1. Area of a figure is a number (in some unit) associated with the part of the plane enclosed by that figure.
2. Two congruent figures have equal areas but the converse need not be true.
3. If X is a planar region formed by two non-overlapping planar regions of figures P and Q, then ar(X) = ar(P) + ar(Q)
4. Two figures are said to be on the same base and between the same parallels, if they have a common base (side) and the vertices (on the vertex) opposite to the common base of each figure lie on a line parallel to the base.
5. Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.
6. Area of a parallelogram is the product of its base and the corresponding altitude.
7. Parallelogram on the same base (or equal bases) and having equal areas lie between the same parallels.
8. If a parallelogram and a triangle are on the same base and between the same parallels, then area of the triangle is half the area of the parallelogram.
9. Triangles on the same base (or equal bases) and between the same parallels are equal in area.
10. Triangles on the same base (or equal bases) and having equal areas lie between the same

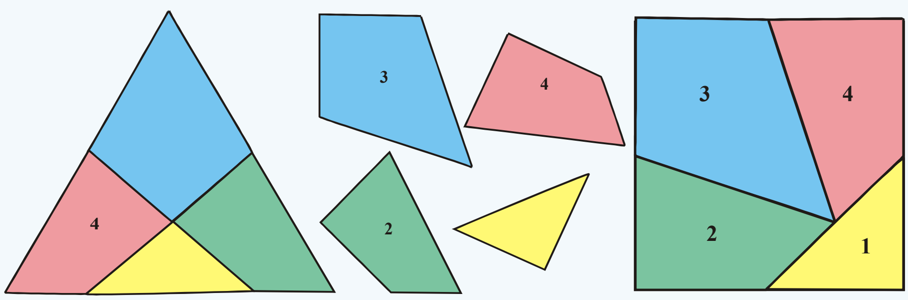
E9C1Q3

Do You Know?

A Puzzle (Areas)

German mathematician David Hilbert (1862-1943) first proved that any polygon can be transformed into any other polygon of equal area by cutting it into a finite number of pieces.

Let us see how an English puzzlist, Henry Ernest Dudency (1847 - 1930) transforms an equilateral triangle into a square by cutting it into four pieces.



Try to make some more puzzles using his ideas and enjoy.

Chapter

12

Circles

H8H8W2

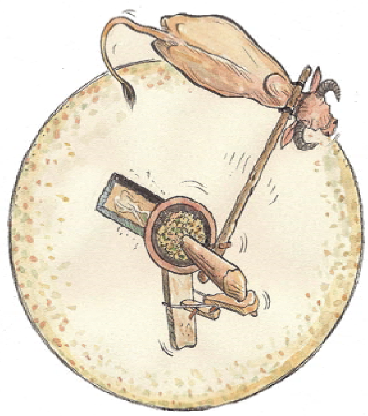
12.1 INTRODUCTION

We come across many round shaped objects in our surroundings such as coins, bangles, clocks, wheels, buttons etc. All these are circular in shape.

You might have drawn an outline along the edges of a coin, a bangle, a button in your childhood to form a circle.

So, can you tell, the difference between the circular objects and the circles you have drawn with the help of these objects?

All the circular objects we have observed above have thickness and are 3-dimensional objects, where as, a circle is a 2-dimensional figure, with no thickness.

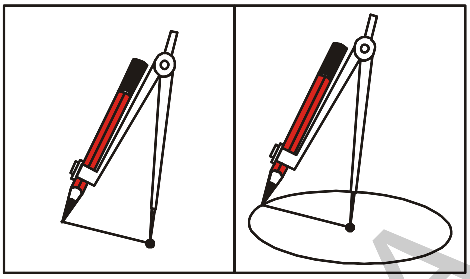


Let us take another example of a circle. You might have seen the oil press called oil mill (Spanish wheel - in Telugu known as ganuga). In the figure, a bullock is tied to fulcrum fixed at a point. Can you identify the shape of the path in which the bullock is moving? It is circular in shape.

A line along the boundary made by the bullock is a circle. The oil press is attached to the ground at a fixed point, which is the centre of the circle. The length of the fulcrum with reference to the circle is radius of the circle. Think of some other examples from your daily life about circles.

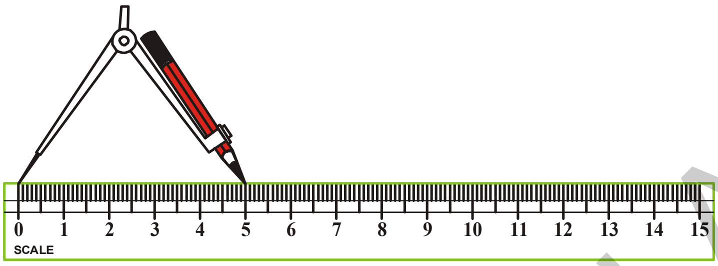
In this chapter we will study circles, related terms and properties of the circle. Before this, you must know how to draw a circle with the help of a compass.

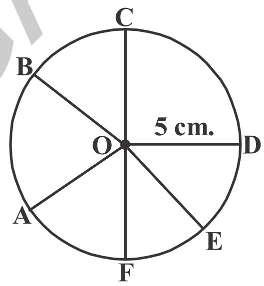
Insert a pencil in the pencil holder of the compass and tighten the screw. Mark a point 'O' on the drawing paper. Fix the sharp point of the compass on 'O'. Keeping the point of the compass firmly move the pencil round on the paper to draw the circle as shown in the figure.



If we need to draw a circle of given radius, we do this with the help of a scale.

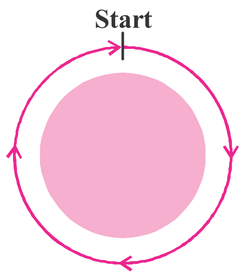
Adjust the distance between the sharp point of the compass and tip of the pencil equal to the length of the given radius, mark a point 'O'(radius of the circle in the figure is 5 cm.) and draw circle as described above.





Mark any 6 points A, B, C, D E and F on the circle. You can see that the length of each line segment OA, OB, OC, OD, OE and OF is 5 cm., which is equal to the given radius. Mark some other points on the circle and measure their distances from the point 'O'. What have you observed? We can say that a circle is a collection of all the points in a plane which are at a fixed distance from a fixed point on the plane.

The fixed point 'O' is called the centre of the circle and the fixed distance OA, is called the **radius** of the circle.

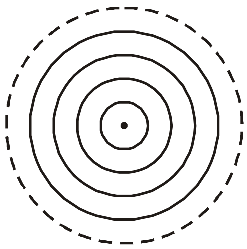


In a circular park Narsimha started walking from a point around the park and completed one round. What do you call the distance covered by Narsimha? It is the total length of the boundary of the circular park, and is called the circumference of the park.

So, the complete length of a circle is called its **circumference**.

Activity

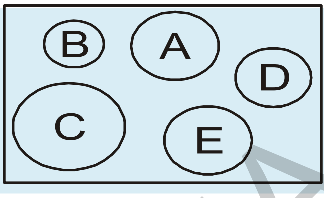
Let us now do the following activity. Mark a point on a sheet of paper. Taking this point as centre draw a circle with any radius. Now increase or decrease the radius and again draw some more circles with the same centre. What do you call the circles obtained in this activity?

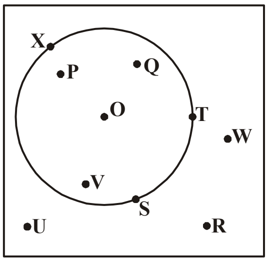


Circles having same centre with different radii are called **concentric circles**.

Do This

1. In the figure, which circles are congruent to the circle A?
2. What measure of the circles make them congruent?





A circle divides the plane on which it lies into three parts. They are (i) inside the circle, which is also called interior of the circle; (ii) on the circle, this is also called the circumference and (iii) outside the circle, which is also called the exterior of the circle. From the above figure, find the points which are inside, outside and on the circle.

The circle and its interior make up the circular region.

Activity

Take a thin circular sheet and fold it to half and open. Again fold it along any other half and open. Repeat this activity for several times. Finally when you open it, what do you observe?

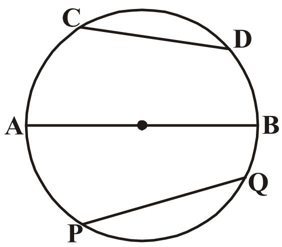
You observe that all creases (traces of the folds) are intersecting at one point. Do you remember what do we call this point? This is the centre of the circle.

Measure the length of each crease of a circle with a divider. What do you notice ? They are all equal and each crease is dividing the circle into two equal halves. That crease is called diameter of circle. Diameter of a circle is twice its radius. A line segment joining any two points on the circle that passes through the centre is called the **diameter**.

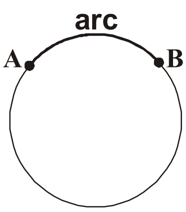
In the above activity if we fold the paper in any manner not only in half, we see that creases joining two points on circle. These creases are called chords of the circle.

So, a line segment joining any two points on the circle is called a **chord.**

What do you call the longest chord? Does it pass through the centre?



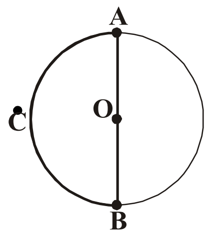
See in the figure, $\overline{CD}$, $\overline{AB}$ and $\overline{PQ}$ are chords of the circle.



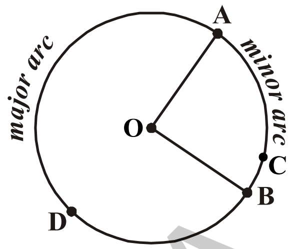
(i)

In the fig.(i), two points A and B are on the circle and they are dividing the circumference of the circle into two parts. The part of the circle between any two points on it is called an arc. In the fig.(i) AB is called an 'arc' and it is denoted by $\overset{\frown}{AB}$. If the end points of an arc become the end points of a diameter then such an arc is called a semicircular arc or a semicircle. In the fig.(ii) $\overset{\frown}{ACB}$ is a semicircle

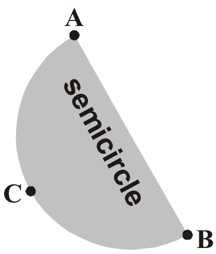
If the arc is smaller than a semicircle, then the arc is called a minor arc and if the arc is longer than a semicircle, then the arc is called a major arc. In the fig.(iii) $\overset{\frown}{ACB}$ is a minor arc and $\overset{\frown}{ADB}$ is a major arc.

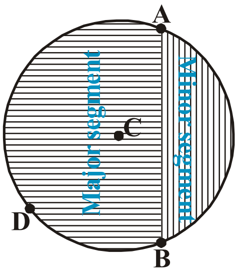


(ii)

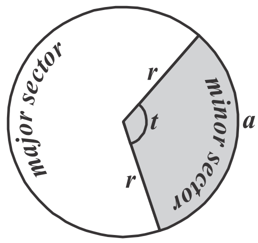


(iii)





If we join the end points of an arc by a chord, the chord divides the circle into two parts. The region between the chord and the minor arc is called the **minor segment** and the region between the chord and the major arc is called the **major segment**. If the chord happens to be a diameter, then the diameter divides the circle into two equal segments.



The region enclosed by an arc and the two radii joining the centre to the end points of an arc is called a ***sector***. One is minor sector and another is major sector (see adjacent figure).

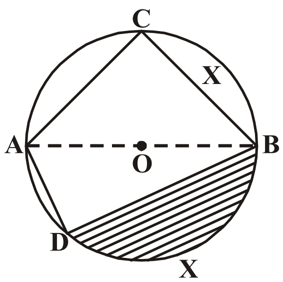
Exercise - 12.1

1. Name the following parts from the adjacent figure where 'O' is the centre of the circle.

(i) $\overline{AO}$ (ii) $\overline{AB}$ (iii) $\overset{\frown}{BC}$

(iv) $\overline{AC}$ (v) $\overset{\frown}{DCB}$ (vi) $\overset{\frown}{ACB}$

(vii) $\overline{AD}$ (viii) shaded region



2. State true or false.

i. A circle divides the plane on which it lies into three parts. ( )

ii. The region enclosed by a chord and the minor arc is minor segment. ( )

iii. The region enclosed by a chord and the major arc is major segment. ( )

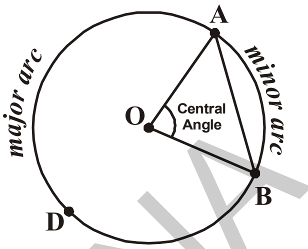
iv. A diameter divides the circle into two unequal parts. ( )

v. A sector is the area enclosed by two radii and a chord ( )

vi. The longest of all chords of a circle is called a diameter. ( )

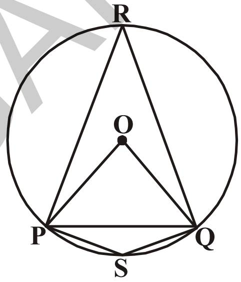
vii. The mid point of any diameter of a circle is the centre. ( )

12.2 Angle subtended by a chord at a point on the Circle



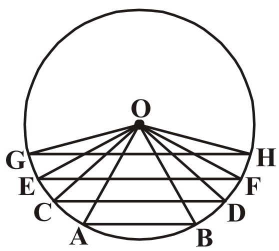
Let A, B be any two points on a circle with centre 'O'. Join A,O and B,O. Angle is made at centre 'O' by $\overline{AO}$, $\overline{BO}$ i.e. $\angle AOB$ is called the angle subtended by the chord $\overline{AB}$ at the centre 'O'.

What do you call the angles $\angle POQ$, $\angle PSQ$ and $\angle PRQ$ in the figure?



i. $\angle POQ$ is the angle subtended by the chord $\overline{PQ}$ at the centre 'O'

ii. $\angle PSQ$ and $\angle PRQ$ are respectively the angles subtended by the chord PQ at point S and point R on the minor and major arc.



In the figure, O is the centre of the circle and $\overline{AB}$, $\overline{CD}$, $\overline{EF}$ and $\overline{GH}$ are the chords of the circle.

What did you observe from the figure?

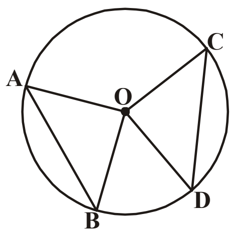
We can observe from the figure that GH > EF > CD > AB.

Now what do you say about the angles subtended by these chords at the centre?

After observing the angles, you will find that the angles subtended by the chords at the centre of the circle increases with increase in the length of chords.

So, now imagine what will happen to the angle subtended at the centre of the circle, if we take two equal chords of a circle?

Construct a circle with centre 'O' and draw equal chords AB and CD using the compass and ruler.



Join the centre 'O' with A, B and with C and D. Now measure the angles $\angle AOB$ and $\angle COD$. Are they equal to each other?

Draw two or more equal chords of a circle and measure the angles subtended by them at the centre.

You will find that the angles subtended by them at the centre are equal.

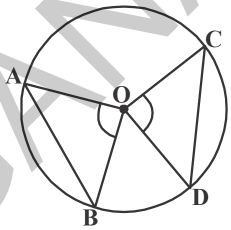
Let us try to prove this fact.

Theorem-12.1 : Equal chords of a circle subtend equal angles at the centre.

Given : Let 'O' be the centre of the circle. $\overline{AB}$ and $\overline{CD}$ are two equal chords and $\angle AOB$ and $\angle COD$ are the angles subtended by the chords at the centre.

R.T.P. : $\angle AOB \cong \angle COD$

Construction : Join the centre to the end points of each chord and you get two triangles ΔAOB and ΔCOD.



Proof: In triangles AOB and COD

$AB = CD$ (given)

$OA = OC$ (radii of same circle)

$OB = OD$ (radii of same circle)

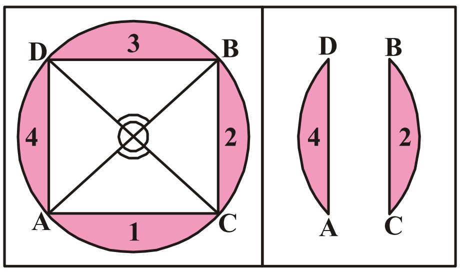
Therefore $\Delta\ AOB \cong \Delta\ COD$ (SSS rule)

Thus $\angle AOB \cong \angle COD$ (corresponding parts of congruent triangles)

In the above theorem, if in a circle, two chords subtend equal angles at the centre, what can you say about the chords? Let us investigate this by the following activity.

Activity

Take a circular paper. Fold it along any diameter such that the two edges coincide with each other. Now open it and again fold it into half along another diameter. On opening, we find two diameters meet at the centre 'O'. There forms two pairs of vertically opposite angles which are equal. Name the end points of the diameter as A, B, C and D.



Draw the chords $\overline{AC}$, $\overline{BC}$, $\overline{BD}$ and $\overline{AD}$.

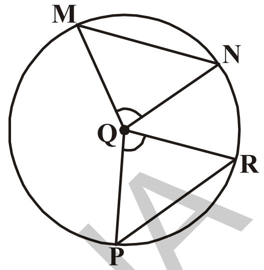
Now take cut-out of the four segments namely 1, 2, 3 and 4.

If you place these segments pair wise one above the other the edges of the pairs (1,3) and (2,4) coincide with each other.

Is $\overline{AD} = \overline{BC}$ and $\overline{AC} = \overline{BD}$?

Though you have seen it in this particular case, try it out for other equal angles too. The chords will all turn out to be equal. We will prove it as a theorem.

Can you state converse of the above theorem (12.1)?



Theorem-12.2 : If the angle subtended by the chords of a circle at the centre are equal, then the chords are equal.

This is the converse of the theorem 12.1.

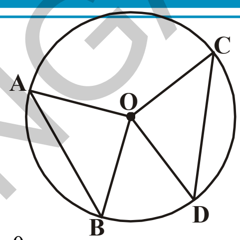
Note that in adjacent figure $\angle PQR = \angle MQN$, then

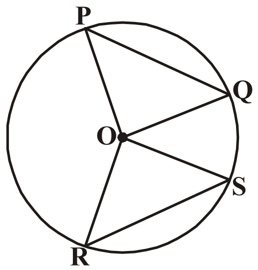
$\Delta PQR \cong \Delta MQN$ (Why?)

Is $PR = MN$? (Verify)

Exercise - 12.2

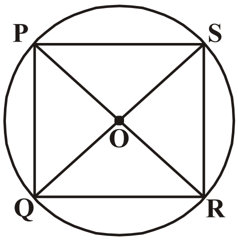
1. In the figure, if $AB = CD$ and $\angle AOB = 90^0$ find $\angle COD$





2. In the figure, $PQ = RS$ and $\angle ORS = 48^0$.

   Find $\angle OPQ$ and $\angle ROS$.

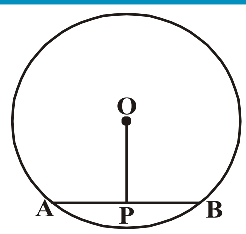


3. In the figure PR and QS are two diameters. Is $PQ = RS$?

12.3 Perpendicular from the Centre to a Chord

Activity

- Construct a circle with centre O. Draw a chord $\overline{AB}$ and a perpendicular to the chord $\overline{AB}$ from the centre 'O'.
- Let the point of intersection of the perpendicular on $\overline{AB}$ be P.
- After measuring PA and PB, we will find $PA = PB$.



Theorem-12.3 : The perpendicular from the centre of a circle to a chord bisects the chord.

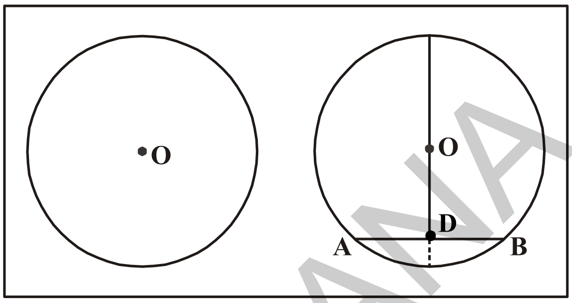
Write a proof by yourself by joining O to A and B and prove that $\Delta OPA \cong \Delta OPB$.

What is the converse of the theorem 12.3?

"If a line drawn from the centre of a circle bisects the chord then the line is perpendicular to that chord"

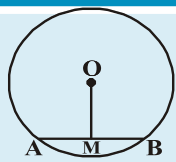
ACTIVITY

Take a circular sheet of paper and mark centre as 'O'



Fold it into two unequal parts and open it. Let the crease represent a chord AB, and then make a fold such that 'A' coincides with B. Mark the point of intersection of the two folds as D. Is AD=DB? and $\angle ODA = \angle ODB$? (Measure the angles between the creases). They are right angles. So, we can make a hypothesis "the line drawn through the centre of a circle to bisect a chord is perpendicular to the chord".

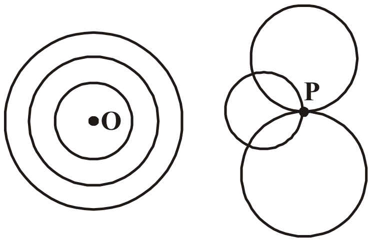
TRY THIS



In a circle with centre 'O'. $\overline{AB}$ is a chord and 'M' is its midpoint. Now prove that $\overline{OM}$ is perpendicular to $\overline{AB}$.

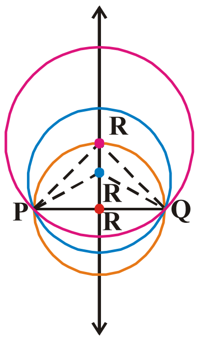
(**Hint** : Join $\overline{OA}$ and $\overline{OB}$ consider triangles OAM and OBM)

12.3.1 The three points that describe a circle



Let 'O' be a point on a plane. How many circles we can draw with centre 'O'? As many circles as we wish. We have already learnt that these circles are called concentric circles. If 'P' is a point other than the centre of the circle, then also we can draw many circles through P.

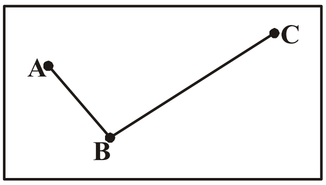
Suppose that there are two distinct points P and Q



How many circles can be drawn passing through given two points? We see that we can draw many circles passing through P and Q.

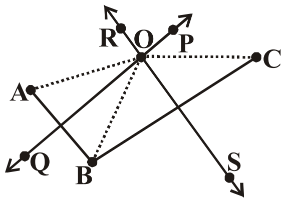
Let us join P and Q, draw the perpendicular bisector to $\overline{PQ}$. Take any three points R, R_1 and R_2 on the perpendicular bisector and draw circles with centre R, R_1, R_2 and radii RP, R_1P and R_2P respectively. Does these circles also passes through Q (Why?)

As every point on the perpendicular bisector of a line segment is equidistant from end points of the line segment. Centre of a circle lies on the perpendicular of any chord.



If three non-collinear points are given, then how many circles can be drawn through them? Let us examine it. Take any three non-collinear points A, B, C and join A, B and B, C.

Draw $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ the perpendicular bisectors to $\overline{AB}$ and $\overline{BC}$. respectively. Both of them intersect at a point 'O'(since two lines cannot have more than one point in common)

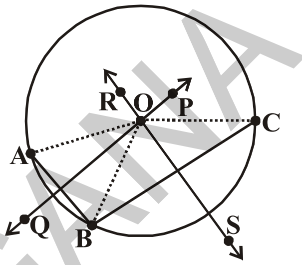


Now O lies on the perpendicular bisector of $\overline{AB}$, so OA = OB. .....(i)

(As every point on $\overleftrightarrow{PQ}$ is at equidistant from A and B)

Also, 'O' lies on the perpendicular bisectors of $\overline{BC}$

Therefore OB = OC ..... (ii)



From equation (i) and (ii)

We can say that OA = OB = OC (transitive law)

Therefore, 'O' is the only point which is equidistant from the points A, B and C so if we draw a circle with centre O and radius OA, it will also pass through B and C i.e. we have only one circle that passes through A, B and C.

The hypothesis based on above observation is "there is one and only one circle that passes through three non-collinear points"

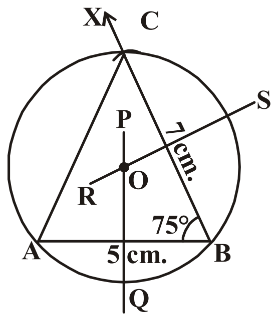
Note **:** If we join AC, the triangle ABC is formed. All its vertices lie on the circle. This circle is called circum circle of the triangle, the centre of the circle 'O' is circumcentre and the radius OA or OB or OC i.e. is circumradius. Generally circumradius is denoted by 'R'.

Try This

If three points are collinear, how many circles can be drawn through these points? Now, try to draw a circle passing through these three points.

Example-1. Construct a circumcircle of the triangle ABC where AB = 5cm; $\angle B = 75°$ and BC = 7cm

Solution : Draw a line segment AB= 5 cm. Draw $\overrightarrow{BX}$ at B such that $\angle B = 75°$. Draw an arc of radius 7cm with centre B to cut $\overrightarrow{BX}$ at C. Join CA to form ΔABC. Draw perpendicular bisectors $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ to $\overline{AB}$ and $\overline{BC}$ respectively. $\overleftrightarrow{PQ}$, $\overleftrightarrow{RS}$ intersect at 'O'. Keeping 'O' as a centre, draw a circle with OA as radius. The circle also passes through B and C and this is the required circumcircle.

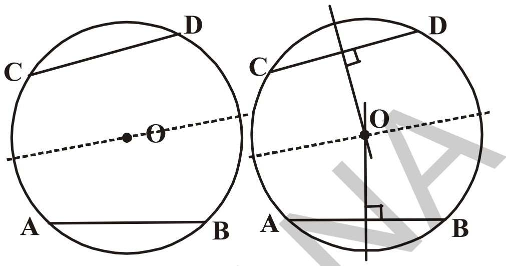


12.3.2 Chords and their distance from the centre of the circle

A circle can have infinite chords. Suppose we make many chords of equal length in a circle, then what would be the distance of these chords of equal length from the centre? Let us examine it through this activity.

ACTIVITY

Draw a big circle on a paper and take a cut-out of it. Mark its centre as ‘O’. Fold it in half. Now make another fold near semi-circular edge. Now unfold it. You will get two conguent folds of chords. Name them as AB and CD. Now make perpendicular folds passing through centre ‘O’ for them. Using divider compare the perpendicular distances of these chords from the centre.



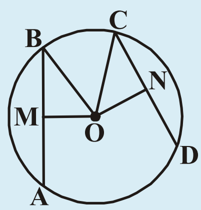
Repeat the above activity by folding congruent chords. State your observations as a hypothesis.

“The congruent chords in a circle are at equal distance from the centre of the circle”

TRY THIS

In the figure, ‘O’ is the centre of the circle and AB = CD. OM is perpendicular on $\overline{AB}$ and $\overline{ON}$ is perpendicular on $\overline{CD}$. Then prove that OM = ON.

As the above hypothesis has been proved logically, it becomes the theorem ‘chords of equal length are at equal distance from the centre of the circle.’



Example-2. In the figure, O is the centre of the circle. Find the length of CD, if AB = 5 cm.

Solution : In Δ AOB and Δ COD,

OA = OC (why?)

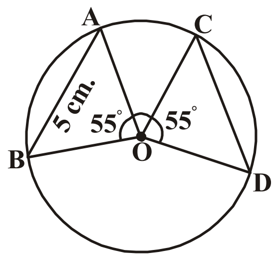
OB = OD (why?)

$\angle$AOB = $\angle$COD

$\therefore$ Δ AOB $\cong$ Δ COD (Criterian of congruent by SAS)

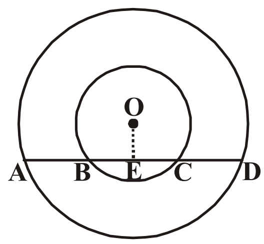
$\because$ AB = CD (Congruent parts of congruent triangles)

$\therefore$ AB = 5 cm. then CD = 5 cm.



Example-3. In the adjacent figure, there are two concentric circles with centre ‘O’. Chord AD of the bigger circle intersects the smaller circle at B and C. Show that AB = CD.

Given : In two concentric circles with centre ‘O’. $\overline{AD}$ is the chord of the



R.T.P. : AB = CD

Construction : Draw $\overline{OE}$ perpendicular to $\overline{AD}$

Proof : AD is the chord of the bigger circle with centre 'O' and $\overline{OE}$ is perpendicular to $\overline{AD}$.

$\because$ $\overline{OE}$ bisects $\overline{AD}$ (The perpendicular from the centre of a circle to a chord bisect it)

$\therefore$ AE = ED ..... (i)

$\overline{BC}$ is the chord of the smaller circle with centre 'O' and $\overline{OE}$ is perpendicular to AD.

$\because$ $\overline{OE}$ bisects $\overline{BC}$ (from the same theorem)

$\therefore$ BE = CE ..... (ii)

Subtracting the equation (ii) from (i), we get

AE - BE = ED - EC

AB = CD

Exercise - 12.3

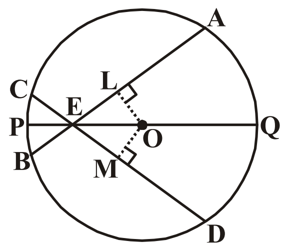
1. Draw the following triangles and construct circumcircles for them.

   (i) In Δ ABC, AB = 6cm, BC = 7cm and $\angle A = 60^{o}$

   (ii) In Δ PQR, PQ = 5cm, QR = 6cm and RP = 8.2cm

   (iii) In Δ XYZ, XY = 4.8cm, $\angle X = 60^{o}$ and $\angle Y = 70^{o}$

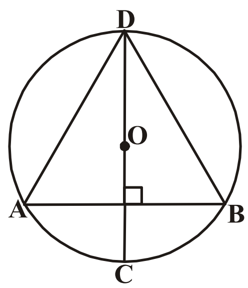
2. Draw two circles passing through A, B where AB = 5.4cm



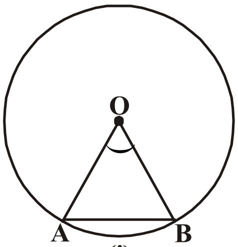
3. If two circles intersect at two points, then prove that their centres lie on the perpendicular bisector of the common chord.

4. If two intersecting chords of a circle make equal angles with diameter passing through their point of intersection, prove that the chords are equal.

5. In the adjacent figure, $\overline{AB}$ is a chord of circle with centre O. $\overline{CD}$ is the diameter perpendicualr to $\overline{AB}$. Show that AD = BD.



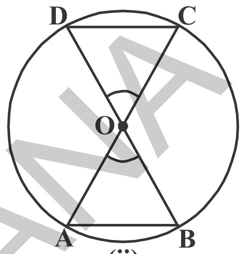
12.4 ANGLE SUBTENDED BY AN ARC OF A CIRCLE



(i)

In the fig.(i), $\overline{AB}$ is a chord and $\overset{\frown}{AB}$ is an arc (minor arc). The end points of the chord and arc are the same i.e. A and B.

Therefore angle subtended by the chord at the centre 'O' is the same as the angle subtended by the arc at the centre 'O'.



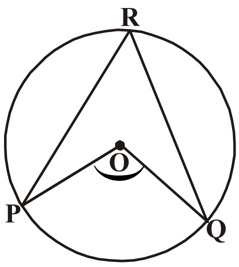
(ii)

In fig.(ii) $\overline{AB}$ and $\overline{CD}$ are two chords of a circle with centre 'O'. If AB = CD, then $\angle AOB = \angle COD$

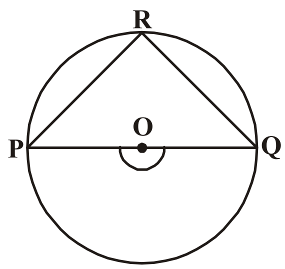
Therefore we can say that the angle subtended by an arc $\overset{\frown}{AB}$ is equal to the angle subtended by the arc $\overset{\frown}{CD}$ at the centre 'O'. (Prove $\Delta AOB \cong \Delta DOC$)

From the above observations we can conclude that in the same cirlce or congruent circles *"arcs of equal length subtend equal angles at the centre"* [$\therefore$ Angle subtended by an arc at the centre is called a measure of that arc]

12.4.1 Angle subtended by an arc at a point on the remaining part of circle



(i)



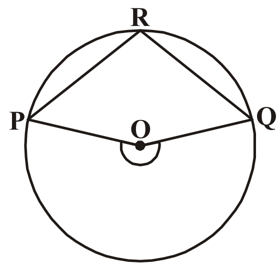
(ii)

Consider the circle with centre 'O'.

Let $\overset{\frown}{PQ}$ in fig. (i) the minor arc, in fig. (ii) semicircle and in fig. (iii) major arc.

Take any point R on the circle. Join R with P and Q.

$\angle PRQ$ is the angle subtended by the arc PQ at the point R on the circle while $\angle POQ$ is subtended at the centre.



(iii)

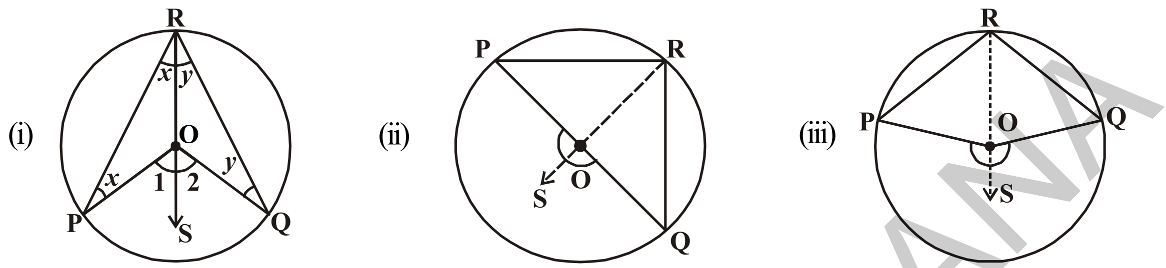
Complete the following table for the given figures.

Angle	Fig. (i)	Fig. (ii)	Fig. (iii)
$\angle PRQ$			
$\angle POQ$			

Similarly draw some circles and subtended angles on the circumference and centre of the circle by their arcs. What do you notice? Can you make a conjecture about the angle made by an arc at the centre and a point on the circle? So from the above observations, we can say that *"The angle subtended by an arc at the centre 'O' is twice the angle subtended by it on the remaining arc of the circle".*

Let us prove this hypothesis as a theorem.

Theorem-12.4 : The angle subtended by an arc at the centre of a circle is double the angle subtended by it at any point on the remaining circle.



Given : Let O be the centre of the circle.

$\overparen{PQ}$ is an arc subtending $\angle POQ$ at the centre.

Let R be a point on the remaining part of the circle (not on $\overparen{PQ}$)

Proof: Here we have three different cases in which (i) $\overparen{PQ}$ is minor arc, (ii) $\overparen{PQ}$ is semicircle and (iii) $\overparen{PQ}$ is a major arc

Let us begin by joining the point R with the centre 'O' and extend it to a point S (in all cases)

For all the cases in Δ ROP

OP = OR (radii of the same circle)

Therefore $\angle ORP = \angle OPR$ (Angles opposite to equal sides of an isosceles triangle are equal).

$\angle POS$ is an exterior angle of Δ ROP (construction)

$\angle POS = \angle ORP + \angle OPR$ or $2\angle ORP$ ..... (1)

($\because$ exterior angle = sum of opp. interior angles)

Similarly for ΔROQ

$\angle SOQ = \angle ORQ + \angle OQR$ or $2\angle ORQ$ ... (2)

($\because$ exterior angle is equal to sum of the opposite interior angles)

From (1) and (2)

$\angle POS + \angle SOQ = 2(\angle ORP + \angle ORQ)$

This is same as $\angle POQ = 2\angle QRP$ ..... (3)

For convenience

Let $\angle ORP = \angle OPR = x$

$\angle POS = \angle 1$

$\angle 1 = x + x = 2x$

Let $\angle ORQ = \angle OQR = y$

$\angle SOQ = \angle 2$

$\angle 2 = y + y = 2y$

Now $\angle POQ = \angle 1 + \angle 2 = 2x + 2y$

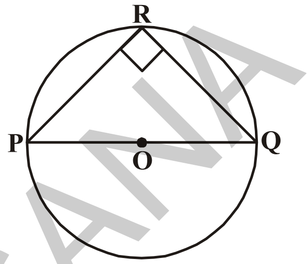
$= 2(x+y) = 2(\angle PRO + \angle ORQ)$

(i.e.) $\angle POQ = 2\angle PRQ$

Hence the theorem is "the angle subtended by an arc at the centre is twice the angle subtended by it at any point on the remaining part of the circle.

Example-4. Let 'O' be the centre of a circle, $\overline{PQ}$ is a diameter, then prove that $\angle PRQ = 90^o$

(OR) Prove that angle in a semi-circle is right angle.



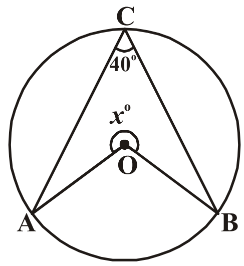
Solution : It is given that $\overline{PQ}$ is a diameter and 'O' is the centre of the circle.

$\therefore \angle POQ = 180^o$ [Angle on a straight line]

and $\angle POQ = 2\angle PRQ$ [Angle subtended by an arc at the centre is twice the angle subtended by it at any other point on circle]

$\therefore \angle PRQ = \dfrac{180^o}{2} = 90^o$

Example-5. Find the value of x° in the adjacent figure



Solution : Given $\angle ACB = 40^\circ$

By the theorem angle made by the arc AB at the centre

$\angle AOB = 2\angle ACB = 2 \times 40^\circ = 80^\circ$

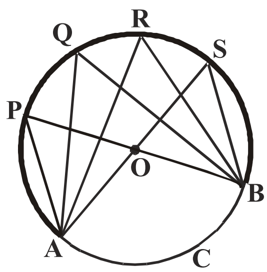
$\because \quad x^\circ + \angle AOB = 360^\circ$

Therefore $x^\circ = 360^\circ - 80^\circ = 280^\circ$

12.4.2 Angles in the same segment

Let us now discuss the measures of angles made by an arc in the same segment of a circle.

Consider a circle with centre 'O' and a minor arc AB (See figure). Let P, Q, R and S be points on the major arc AB i.e. on the remaining part of the circle. Now join the end points of the arc AB with points P, Q, R and S to form angles $\angle APB$, $\angle AQB$, $\angle ARB$ and $\angle ASB$.



$\because \quad \angle AOB = 2\angle APB$ (why?)

$\angle AOB = 2\angle AQB$ (why?)

$\angle AOB = 2\angle ARB$ (why?)

$\angle AOB = 2\angle ASB$ (why?)

Therefore $\angle APB = \angle AQB = \angle ARB = \angle ASB$

Observe that "angles subtended by an arc in the same segment are equal".

Note : In the above discussion we have seen that the point P, Q, R, S and A, B lie on the same circle. What do you call them? "Points lying on the same circle are called concyclic".

The converse of the above theorem can be stated as follows-

Theorem-12.5 : If a line segment joining two points, subtends equal angles at two other points lying on the same side of the line then these, the four points lie on a circle (i.e. they are concyclic)

Given : Two angles $\angle ACB$ and $\angle ADB$ are on the same side of a line segment $\overline{AB}$ joining two points A and B are equal.

R.T.P : A, B, C and D are concyclic (i.e.) they lie on the same circle.

Construction : Draw a circle passing through the three non colinear point A, B and C.

Proof: Suppose the point 'D' does not lie on the Circle.

Then there may be other point 'E' such that it will intersect AD (or extension of AD)

If points A, B, C and E lie on the circle then

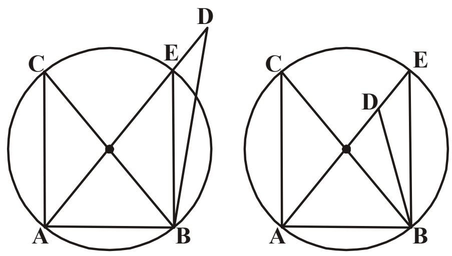
$\angle ACB = \angle AEB$ (Why?)

But it is given that $\angle ACB = \angle ADB$.

Therefore $\angle AEB = \angle ADB$

This is not possible unless E coincides with D (Why?)

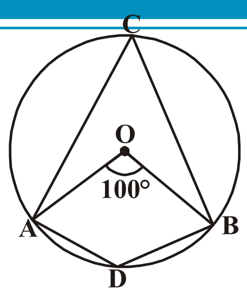
Therefore, E coincides with D.

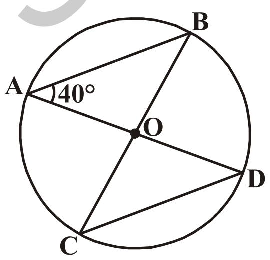


EXERCISE – 12.4

1. In the figure, 'O' is the centre of the circle.

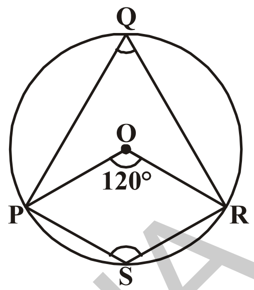
   $\angle AOB = 100^\circ$ find $\angle ADB$.

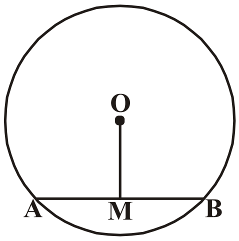




2. In the figure, $\angle BAD = 40^\circ$ then find $\angle BCD$.

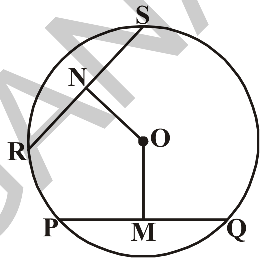
3. In the figure, O is the centre of the circle and $\angle POR = 120^{\circ}$. Find $\angle PQR$ and $\angle PSR$

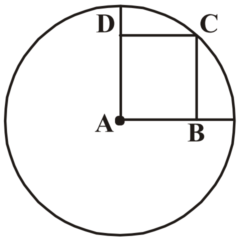




4. In the figure, 'O' is the centre of the circle. OM = 3cm and AB = 8cm. Find the radius of the circle

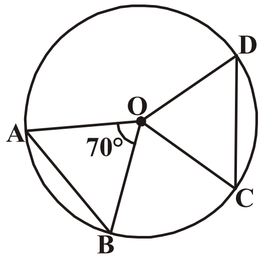
5. In the figure, 'O' is the centre of the circle and $\overline{OM}$, $\overline{ON}$ are the perpendiculars from the centre to the chords $\overline{PQ}$ and $\overline{RS}$. If OM = ON and PQ = 6cm. Find RS.





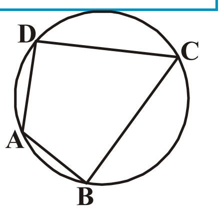
6. A is the centre of the circle and ABCD is a square. If BD = 4cm then find the radius of the circle.

7. Draw a circle with any radius and then draw two chords equidistant from the centre.



8. In the given figure 'O' is the centre of the circle and $\overline{AB}$, $\overline{CD}$ are equal chords. If $\angle AOB = 70^{\circ}$. Find the angles of the ΔOCD.

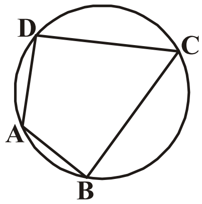
12.5 Cyclic Quadrilateral



In the figure, the vertices of the quadrilateral A, B, C and D lie on the same circle, this type of quadrilateral ABCD is called cyclic quadrilateral.

Activity

Draw a circle. Mark four points A, B, C and D on it. Draw quadrilateral ABCD. Measure its angles. Record them in the table. Repeat this activity for three more times.



S.No	$\angle A$	$\angle B$	$\angle C$	$\angle D$	$\angle A+\angle C$	$\angle B+\angle D$
1						
2						
3						
4						

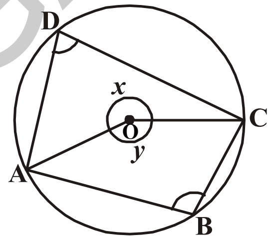
What do you infer from the table?

Theorem-12.6 : The opposite angles of a cyclic quadrilateral are supplementary.

Given : ABCD is a cyclic quadrilateral .

To Prove : $\angle A + \angle C = 180°$

$\angle B + \angle D = 180°$



Construction : Join OA, OC

Proof: $\angle D = \frac{1}{2}\angle y$ (Why?) ..... (i)

$\angle B = \frac{1}{2}\angle x$ (Why?) ..... (ii)

By adding of (i) and (ii)

$$\angle D + \angle B = \frac{1}{2}\angle y + \frac{1}{2}\angle x$$

$$\angle D + \angle B = \frac{1}{2}(\angle y + \angle x)$$

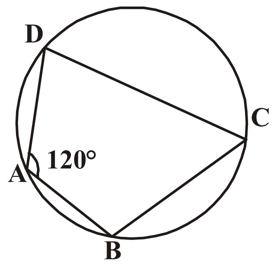
$$\angle B + \angle D = \frac{1}{2} \times 360°$$

$$\angle B + \angle D = 180°$$

Similarly $\angle A + \angle C = 180°$

Example-6. In the figure, $\angle A = 120^{\circ}$ then find $\angle C$

Solution: ABCD is a cyclic quadrilateral



Therefore $\angle A + \angle C = 180^{\circ}$

$120^0 + \angle C = 180^{\circ}$

Therefore $\angle C = 180^{\circ} - 120^{\circ} = 60^{\circ}$

What is the converse of the above theorem?

"If the sum of a pair of opposite angles of a quadrilateral is 180^0, then the quadrilateral is cyclic".

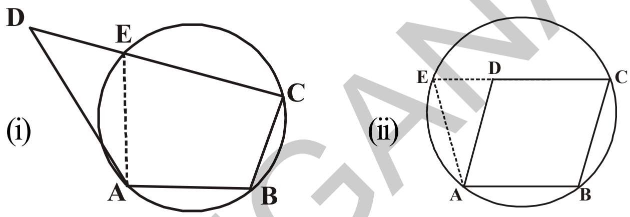
The converse is also true.

Theorem-12.7 : If the sum of any pair of opposite angles in a quadrilateral is 180^o, then it is cyclic.

Given : Let ABCD be a quadrilateral such that

$\angle ABC + \angle ADC = 180^o$

$\angle DAB + \angle BCD = 180^o$



R.T.P. : ABCD is a cyclic quadrilateral.

Construction : Draw a circle through three non-collinear points A, B, and C.

If it passes through D, the theorem is proved since A, B, C and D are concyclic. If the circle does not pass through D, it intersects $\overline{CD}$ [fig (i)] or $\overline{CD}$ produced [fig (ii)] at E.

Draw $\overline{AE}$

Proof : ABCE is a cyclic quadrilateral (construction)

$\angle AEC + \angle ABC = 180^o$ [sum of the opposite angles of a cyclic quadrilateral]

But $\angle ABC + \angle ADC = 180^o$ (Given)

$\therefore \angle AEC + \angle ABC = \angle ABC + \angle ADC \Rightarrow \angle AEC = \angle ADC$

But one of these is an exterior angle of ΔADE and the other is an interior opposite angle.

We know that the exterior angle of a triangle is always greater than either of the opposite interior angles.

$\therefore \angle AEC = \angle ADC$ is a contradiction.

So our assumption that the circle passing through A, B and C does not pass through D is false.

$\therefore$ The circle passing through A, B, C also passes through D.

$\therefore$ A, B, C and D are concyclic. Hence ABCD is a cyclic quadrilateral.

Example-7. In figure, $\overline{AB}$ is a diameter of the circle, $\overline{CD}$ is a chord equal to the radius of the circle. $\overrightarrow{AC}$ and $\overrightarrow{BD}$ when extended intersect at a point E. Prove that $\angle AEB = 60°$.

Solution : Join O,C, join O,D and join B,C.

Triangle ODC is equilateral (Why?)

Therefore, $\angle COD = 60°$

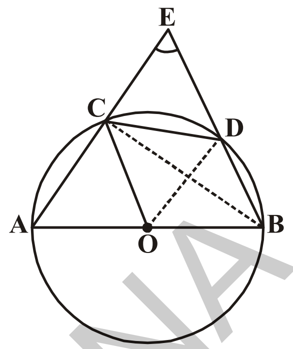
Now, $\angle CBD = \frac{1}{2} \angle COD$ (Why?)

This gives $\angle CBD = 30°$

Again, $\angle ACB = 90°$ (Why?)

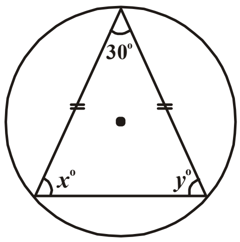
So, $\angle BCE = 180° - \angle ACB = 90°$

Which gives $\angle CEB = 90° - 30° = 60°$, i.e. $\angle AEB = 60°$

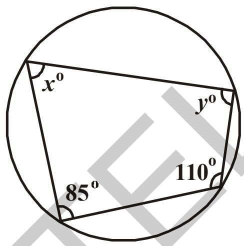


EXERCISE 12.5

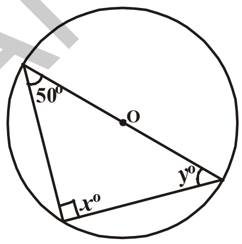
1. Find the values of x and y in the figures given below.



(i)



(ii)



(iii)

2. Given that the vertices A, B, C of a quadrilateral ABCD lie on a circle.

   Also $\angle A + \angle C = 180°$, then prove that the vertex D also lie on the same circle.

3. If a parallelogram is cyclic, then prove that it is a rectangle.

4. Prove that a cyclic rhombus is a square.

5. For each of the following, draw a circle and inscribe the figure given. If a polygon of the given type can't be inscribed, write not possible.

   (a) Rectangle

   (b) Trapezium

   (c) Obtuse triangle

   (d) Non-rectangular parallelogram

   (e) Accute isosceles triangle

   (f) A quadrilateral PQRS with $\overline{PR}$ as diameter.

What we have discussed?

- A collection of all points in a plane which are at a fixed distance from a fixed point in the same plane is called a circle. The fixed point is called the centre and the fixed distance is called the radius of the circle
- A line segment joining any points on the circle is called a chord
- The longest of all chords which also passes through the centre is called a diameter.
- Circles with same radii are called congruent circles
- Circles with same centre and different radii are called concentric circles
- Diameter of a circle divides it into two semi-circles
- The part between any two points on the circle is called an arc
- The area enclosed by a chord and an arc is called a segment. If the arc is a minor arc then it is called the minor segment and if the arc is major arc then it is called the major segment
- The area enclosed by an arc and the two radii joining the end points of the arc with centre is called a sector
- Equal chords of a circle subtend equal angles at the centre
- Angles in the same segment are equal
- An angle in a semi circle is a right angle.
- If the angles subtended by two chords at the centre are equal, then the chords are congruent
- The perpendicular from the centre of a circle to a chord bisects the chords. The converse is also true
- There is exactly one circle passes through three non-collinear points
- The circle passing through the vertices of a triangle is called a circumcircle
- Equal chords are at equal distance from the centre of the circle, conversely chords at equidistant from the centre of the circle are equal in length
- Angle subtended by an arc at the centre of the circle is twice the angle subtended by it at any other point on the circle.
- If the angle subtended by an arc at a point on the remaining part of the circle is 90^o, then the arc is a semi circle.
- If a line segment joining two points subtends same angles at two other points lying on the same side of the line segment, the four points lie on the circle.
- The pairs of opposite angles of a cyclic quadrilateral are supplementary.

G7P3R6

Chapter

13 Geometrical Constructions

W3G5M1

13.1 Introduction

To construct geometrical figures, such as a line segment, an angle, a triangle, a quadrilateral etc., some basic geometrical instruments are needed. You must be having a geometry box which contains a graduated ruler (Scale) a pair of set squares, a divider, a compass and a protractor.

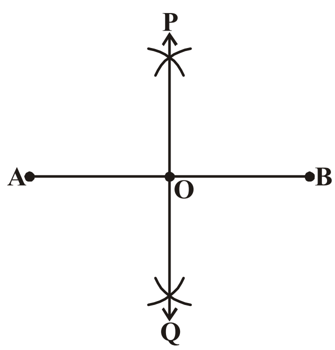
Generally, all these instruments are needed in drawing. A geometrical construction is the process of drawing a geometrical figure using only two instruments - an ungraduated ruler and a compass. We have mostly used ruler and compass in the construction of triangles and quadrilaterals in the earlier classes. In construction where some other instruments are also required, you may use a graduated scale and protractor as well. There are some constructions that cannot be done straight away. For example, when there are 3 measures available for the triangle, they may not be used directly. We will see in this chapter, how to extract the needed values and complete the required shape.

13.2 Basic Constructions

You have learnt how to construct (i) the perpendicular bisector of a line segment, (ii) angle bisector of 30°, 45°, 60°, 90° and 120° or of a given angle, in the lower classes. However the reason for these constructions were not discussed. The objective of this chapter is to give the process of necessary logical proofs to all those constructions.

13.2.1 To Construct the perpendicular bisector of a given line segment.

Example-1. Draw the perpendicular bisector of a given line segment AB and write justification.



Solution : Steps of construction.

Steps 1 : Draw the line segment AB

Step 2 : Taking A as a centre and with radius more than $\frac{1}{2}$ AB, draw an arc on either side of the line segment AB.

Step 3 : Taking 'B' as centre, with the same radius as above, draw arcs so that they intersect the previously drawn arcs.

Step 4 : Mark these points of intersection as P and Q.   Join P and Q.

Step 5 : Let PQ intersect $\overline{AB}$ at the point O

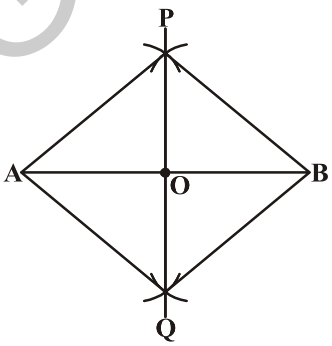
Thus the line POQ is the required perpendicular bisector of AB.

How can you justify the above construction made i.e. "PQ is the perpendicular bisector of AB".

Draw diagram of construction and join A to P and A to Q; also B to P and B to Q.

We use the congruency of triangle properties to prove the required.

Proof :



Steps	**Reasons**
In Δ^s PAQ and ΔPBQ	(Selected triangle)
AP = BP ; AQ = BQ	(Equal radii)
PQ = PQ	(Common side)
$\therefore$ ΔPAQ $\cong$ ΔPBQ	(SSS rule)
So $\angle APO = \angle BPO$	(CPCT (corresponding parts of congruent triangles))
Now In Δ^s APO and BPO	
AP = BP	(Equal radii)
$\angle APO = \angle BPO$	(Proved above)
OP = OP	(Common side)
$\therefore$ ΔAPO $\cong$ ΔBPO	(SAS rule)
So OA = OB and $\angle APO = \angle BPO$	(CPCT)
As $\angle AOP + \angle BOP = 180^\circ$	(Linear pair)
and $\angle APO = \angle BPO$	
We get $\angle AOP = \angle BOP = \dfrac{180^\circ}{2} = 90^\circ$	(From the above result)

Thus PO, i.e. $\overleftrightarrow{POQ}$ is the perpendicular bisector of AB.

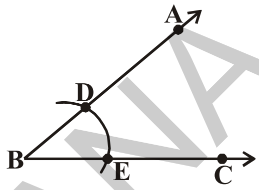
Hence proved.

13.2.2 To construct the bisector of a given angle

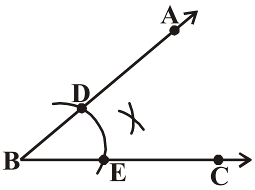
Example-2. Construct the bisector of a given angle $\angle ABC$.

Solution : Steps of construction.

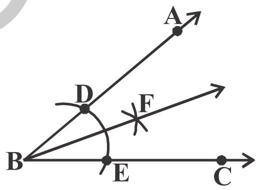
Step 1 : Draw the given angle $\angle ABC$



Step 2 : Taking B as centre and with any radius, draw an arc to intersect the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, at D and E respectively, as shown in the figure.



Step 3 : Taking E and D as centres draw two arcs with equal radii to intersect each other. Let the point of intersection be F.

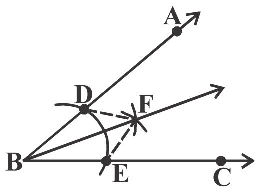


Step 4 : Draw the ray BF. It is the required bisector of $\angle ABC$.

Let us see the logical proof of above construction. Join D, F and E, F. (We use congruency rule of triangles to prove the required).

Proof :

Steps	**Reasons**
In Δ^s BDF and ΔBEF	(Selected triangles)
BD = BE	(radii of same arc)
DF = EF	(Arcs of equal radii)
BF = BF	(Common side)
$\therefore \Delta BDF \cong \Delta BEF$	(SSS rule)
So $\angle DBF = \angle EBF$	(CPCT)



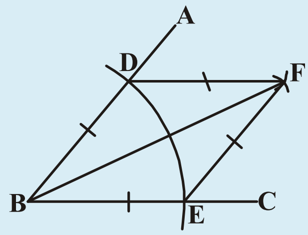
Thus BF is the bisector of $\angle ABC$

$\therefore$ Hence proved.

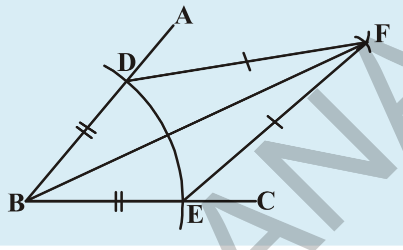
Try These

Observe the sides, angles and diagonals of quadrilateral BEFD. What type of figures are given below and write properties of figures.

1.



2.

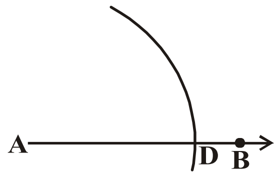


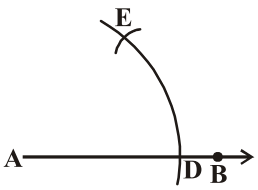
13.2.3 To construct an angle of 60° at the initial point of a given ray.

Example-3. Draw a ray AB (with initial point A) and construct a ray AC such that $\angle BAC = 60°$.

Solution : Steps of Construction

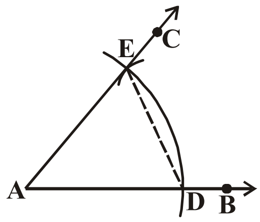
Step 1 : Draw the given ray AB and taking A as centre and some radius, draw an arc which intersects AB, say at a point D.





Step 2 : Taking D as centre and with the same radius taken before, draw an arc intersecting the previously drawn arc, say at a point E (as shown in the fig.)

Step 3 : Draw a ray $\overrightarrow{AC}$ Passing through E then $\angle BAC$ is the required angle of 60°.



Let us see how the construction is justified. Draw the figure again and join D,E and prove it as follows .

Steps	Reasons
In Δ ADE	
AE = AD	(radii of same arc)
AD = DE	(Arcs of equal radius)
Then AE = AD = DE	(Same arc with same radii)
$\therefore$ ΔADE is equilateral triangle	(All sides are equal)
$\angle EAD = 60°$	(each angle of equilateral triangle)
$\angle BAC = \angle EAD$	($\angle EAD$ is a part of $\angle BAC$)
$\angle BAC = 60^o$.	Hence proved.

Try This

Draw a circle, Identify a point on it. Cut arcs on the circle with the length of the radius in succession. How many parts can the circle be divided into? Give reason. What will be the length of the choad?

Exercise - 13.1

1. Construct the following angles at the initial point of a given ray and justify the construction.
   (a) 90^o (b) 45^o
2. Construct the following angles using ruler and compass and verify by measuring them by a protractor.
   (a) 30^o (b) $22\frac{1}{2}^o$ (c) 15^o
   (d) 75^o (e) 105^o f) 135^o
3. Construct an equilateral triangle, given its side of length of 4.5 cm and justify the construction.
4. Construct an isosceles triangle, given its base and base angle and justify the construction.
   [Hint : You can take any measure of side and angle]

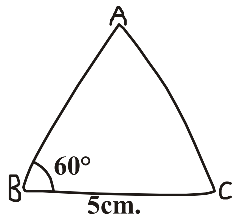
13.3 Construction of triangles (Special cases)

We have so far, constructed some basic constructions and justified with proofs. Now we will construct some triangles when special type of measures are given. Recall the congruency properties of triangles such as SAS, SSS, ASA and RHS rules. You have already learnt how to construct triangles in class VII using the above rules.

You may have learnt that at least three parts of a triangle have to be given for constructing it but not any combinations of three measures are sufficient for the purpose. For example, if two sides and an angle (not the included angle) are given, then it is not always possible to construct such a triangle uniquely. We can give several illustrations for such constructions. In such cases we have to use the given measures with desired combinations such as SAS, SSS, ASA and RHS rules.

13.3.1 Construction : To construct a triangle, given its base, a base angle and sum of other two sides.

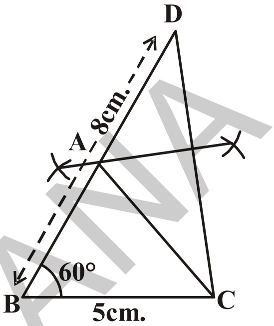
Example-4. Construct a ΔABC given BC = 5 cm., AB + AC = 8 cm. and $\angle ABC = 60^\circ$.



Solution : Steps of construction

Step 1 : Draw a rough sketch of ΔABC and mark the given measurements as usual.
(How can you mark $AB + AC = 8cm$?)
How can you locate third vertex A in the construction ?

Analysis : As we have $AB + AC = 8$ cm., extend BA up to D so that $BD = 8$ cm.



$\therefore\ BD = BA + AD = 8$ cm

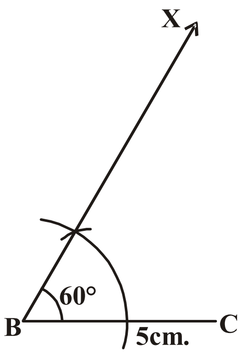
but $AB + AC = 8$ cm. (given)

$\therefore\ AD = AC$

To locate A on BD what will you do ?

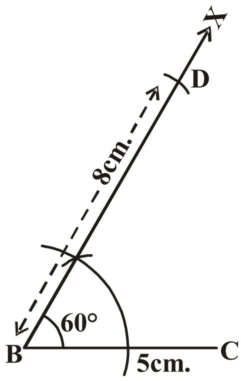
As A is equidistant from C and D, draw a perpendicular bisector of $\overline{CD}$ to locate A on $\overline{BD}$.

How can you prove $AB + AC = BD$?

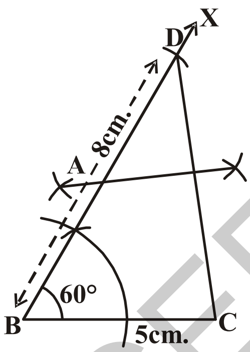


Step 2 : Draw the base BC= 5 cm and construct

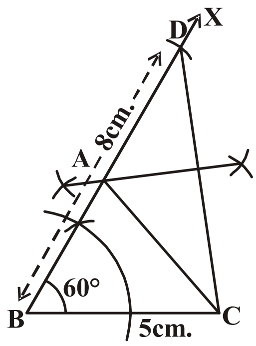
$\angle CBX = 60^\circ$ at B



Step 3: With centre B and radius 8 cm ($AB + AC = 8$ cm) draw an arc on $\overrightarrow{BX}$ to intersect (meet) at D.



Step 4 : Join C, D and draw a perpendicular bisector of CD to meet BD at A



Step 5 : Join A,C to get the required triangle ABC.

Now, we will justify the construction.

Proof : A lies on the perpendicular bisector of $\overline{CD}$

$\because\ AC = AD$

$AB + AC = AB + AD$

$= BD$

$= 8$ cm.

Hence ΔABC is the required triangle.

Think, Discuss and Write

Can you construct a triangle ABC with BC = 6 cm, $\angle B = 60^\circ$ and $AB + AC = 5cm$.? If not, give reasons.

13.3.2 Construction : To Construct a triangle given its base, a base angle and the difference of the other two sides.

Given the base BC of a triangle ABC, a base angle say $\angle B$ and the difference of other two sides $AB - AC$ in case AB>AC or AC-AB, in case AB<AC, you have to construct the triangle ABC. Thus we have two cases of constructions discussed in the following examples.

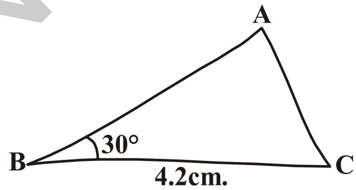
Case (i) Let $AB > AC$

Example-5. Construct ΔABC in which $BC = 4.2$ cm, $\angle B = 30^\circ$ and $AB - AC = 1.6$ cm

Solution : Steps of Construction

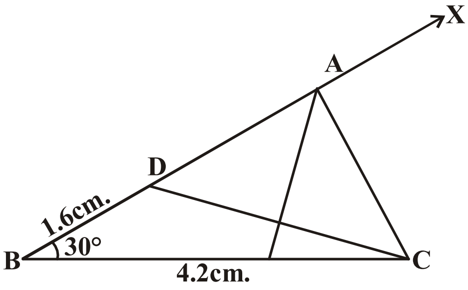
Step 1: Draw a rough sketch of ΔABC and mark the given measurements

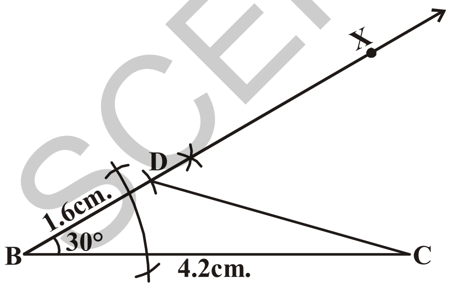
(How can you mark $AB - AC = 1.6$ cm ?)



Analysis : Since $AB - AC = 1.6$ cm and $AB > AC$, mark D on AB such that AD = AC Now $BD = AB - AC = 1.6$ cm. Join CD and draw a perpendicular bisector of CD to find the vertex A on BD produced.

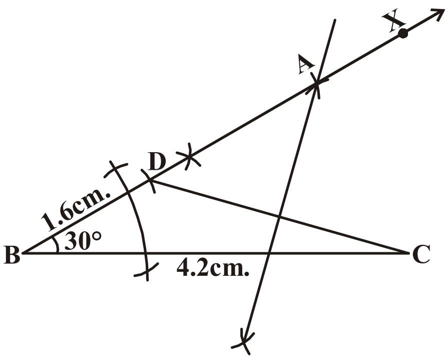
Join AC to get the required triangle ABC.



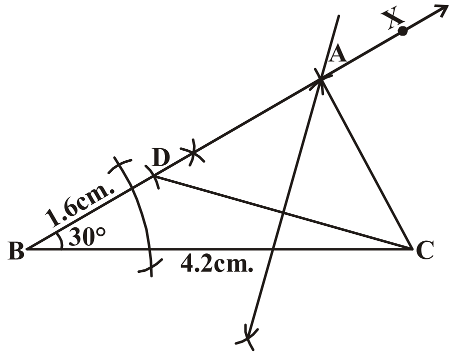


Step 2: Construct ΔBCD using SAS rule with measures $BC = 4.2$ cm $\angle B = 30^0$ and BD=1.6 cm. (i.e. AB - AC)

Step 3 : Draw the perpendicular bisector of CD. Let it meet ray BDX at a point A.



Step 4: Join AC to get the required triangle ABC.



THINK, DISCUSS AND WRITE

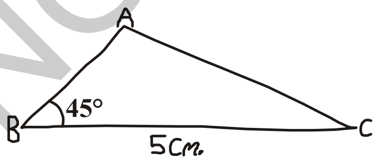
Can you construct the triangle ABC with the same measures by changing the base angle $\angle C$ instead of $\angle B$? Draw a rough sketch and construct it.

Case (ii) Let AB < AC

Example-6. Construct ΔABC in which BC = 5cm, $\angle B = 45^o$ and AC – AB = 1.8 cm.

Solution : Steps of Construction.

Step 1: Draw a rough sketch of ΔABC and mark the given measurements.



Analyse how AC – AB = 1.8 cm can be marked?

Analysis : Since AC – AB = 1.8 cm i.e. AB < AC we have to find D on AB produced such that AD = AC

Now BD = AC – AB = 1.8 cm ($\because$ BD = AD – AB and AD = AC)

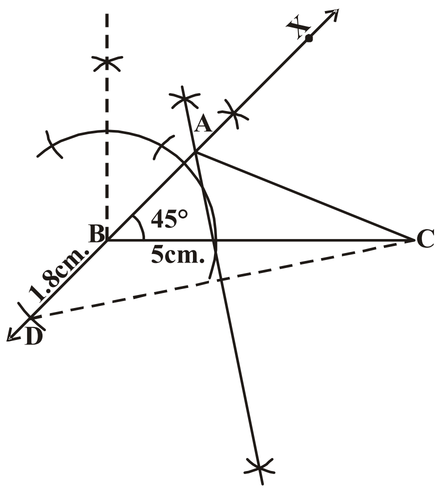
Join CD to find A on the perpendicular bisector of DC

Step 2 : Draw BC = 5 cm and construct $\overrightarrow{BX}$ such that $\angle CBX = 45°$

With centre B and radius 1.8 cm (BD = AC – AB) draw an arc to intersect the line XB extended at a point D.

Step 3 : Join D, C and draw the perpendicular bisector of DC.

Step 4 : Let it meet $\overrightarrow{BX}$ at A and join A, C ΔABC is the required triangle.



Now, you can justify the construction.

Proof: In ΔABC, the point A lies on the perpendicular bisector of $\overline{DC}$.

$\therefore$ AD = AC

AB + BD = AC

So BD = AC – AB

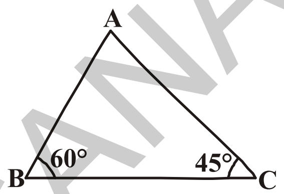
= 1.8 cm

Hence ΔABC is the required that triangle.

13.3.3 Construction : To construct a triangle, given its perimeter and its two base angles.

Given the base angles, say $\angle B$ and $\angle C$ and perimeter AB + BC + CA, you have to construct the triangle ABC.

Example-7. Construct a triangle ABC, in which $\angle B = 60^\circ$, $\angle C = 45^\circ$ and AB + BC + CA = 11 cm.



Solution : Steps of construction.

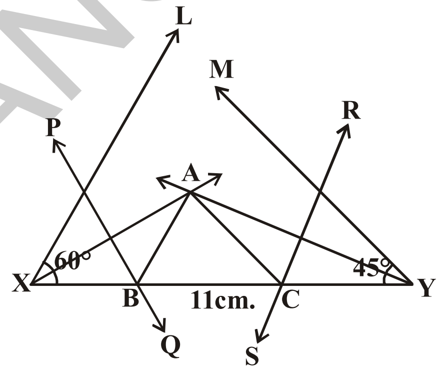
Step 1 : Draw a rough sketch of a triangle ABC and mark the given measures

(Can you mark the perimeter of triangle ?)

Analysis : Draw a line segment, say XY equal to perimeter of ΔABC i.e., AB + BC + CA. Make angles $\angle YXL$ equal to $\angle B$ and $\angle XYM$ equal to $\angle C$ and bisect them.

Let these bisectors intersect at a point A.

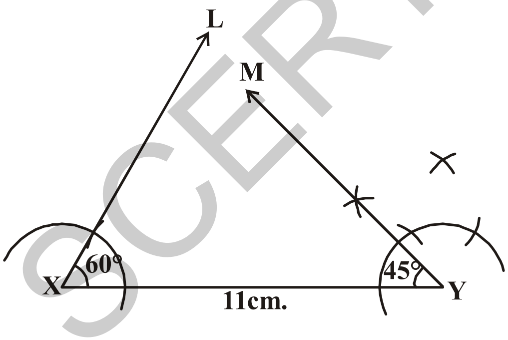
Draw perpendicular bisectors of $\overrightarrow{AX}$ to intersect XY at B and the perpendicular bisector of $\overrightarrow{AY}$ to intersect it at C. Then by joining AB and AC, we get required triangle ABC.



Step 2: Draw a line segment XY = 11 cm

(As XY = AB + BC + CA)

Step 3 : Construct $\angle YXL = 60^\circ$ and $\angle XYM = 45^\circ$ and draw bisectors of these angles.

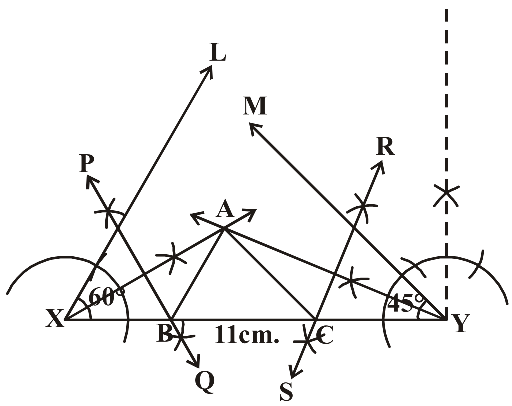


Step 4 : Let the bisectors of these angles intersect at a point A and join AX or AY.

Step 5 : Draw perpendicular bisectors of $\overline{AX}$ and $\overline{AY}$ to intersect $\overrightarrow{XY}$ at B and C respectively

Join A, B and A, C.

Then, ABC is the required triangle.



You can justify the construction as follows

Proof: B lies on the perpendicular bisector PQ of AX

$\therefore$ XB = AB and similarly CY = AC

This gives AB + BC + CA = XB + BC + CY

= XY

Again $\angle BAX = \angle AXB$ ($\because$ XB = AB in ΔAXB) and

$\angle ABC = \angle BAX + \angle AXB$

(Exterior angle of ΔABC).

$= 2\angle AXB$

$= \angle YXL$

$= 60^o$.

Similarly $\angle ACB = \angle XYM = 45^o$ as required

$\therefore$ $\angle B = 60°$ and $\angle C = 45°$ as given are constructed.

TRY THESE

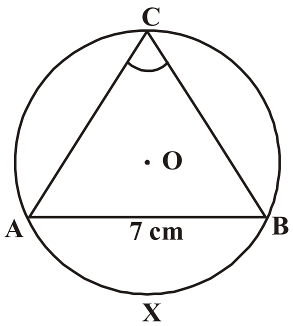
Can you draw the triangle with the same measurements in alternate way?

(**Hint**: Take $\angle YXL = \frac{60°}{2} = 30°$

and $\angle XYM = \frac{45°}{2} = 22\frac{1}{2}°$)

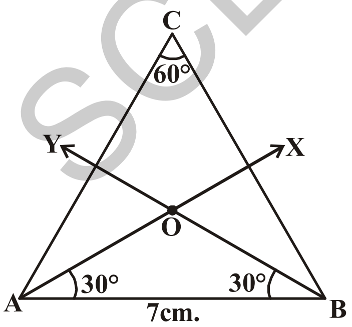
13.3.4 Construction : To construct a circle segment given a chord and a given an angle.

Example-8. Construct a segment of a circle on a chord of length 7cm. and containing an angle of 60°.



Solution : Steps of construction.

Step-1: Draw a rough sketch of a circle and a segment contains an angle 60°. (Draw major segment Why?) Can you draw a circle without a centre?



Analysis: Let 'O' be the centre of the circle. Let AB be the given chord and ACB be the required segment of the circle containing an angle C = 60°.

Let $\overparen{AXB}$ be the arc subtending the angle 60° at C.

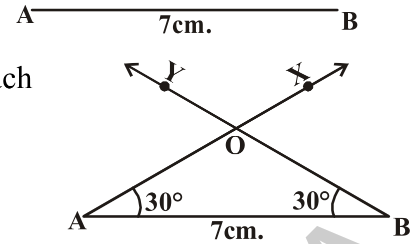
Since $\angle ACB = 60°$, $\angle AOB = 60° \times 2 = 120°$ (How?)

In ΔOAB, OA=OB (radii of same circle)

$$\therefore \angle OAB = \angle OBA = \frac{180° - 120°}{2} = \frac{60°}{2} = 30°$$

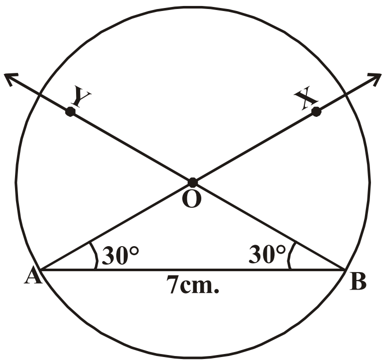
So we can draw ΔOAB and then draw a circle with radius equal to OA or OB.

Step-2 : Draw a line segment AB = 7cm.



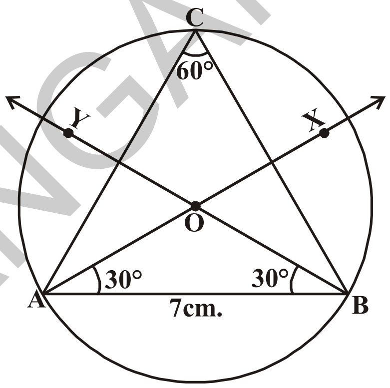
Step-3 : Draw $\overrightarrow{AX}$ such that $\angle BAX = 30°$ and draw $\overrightarrow{BY}$ such that $\angle YBA = 30°$ to intersect $\overrightarrow{AX}$ at O .

[Hint : Construct 30^o angle by bisecting 60^o angle]



Step-4 : With centre 'O' and radius OA or OB, draw the circle.

Step-5 : Mark a point 'C' on the arc of the circle. Join A, C and B, C. We get $\angle ACB = 60°$



Thus ACB is the required circle segment.

Let us justify the construction

Proof : OA = OB (radii of circle).

$\therefore \quad \angle OAB + \angle OBA = 30° + 30° = 60°$

$\therefore \quad \angle AOB = 180° - 60° = 120°$

$\overset{\frown}{AXB}$ Subtends an angle of 120° at the centre of the circle.

$$\therefore \quad \angle ACB = \frac{120°}{2} = 60°$$

$\therefore$ ACB is the required segment of a circle.

Try This

What happens if the angle in the circle segment is right angle? What kind of segment do you obtain? Draw the figure and give reason.

Exercise - 13.2

1. Construct ΔABC in which BC = 7 cm, $\angle B = 75^{\circ}$ and AB + AC = 12 cm.
2. Construct ΔPQR in which QR = 8 cm, $\angle Q = 60°$ and PQ – PR = 3.5 cm
3. Construct Δ XYZ in which $\angle Y = 30^{\circ}$, $\angle Z = 60^{\circ}$ and XY + YZ + ZX = 10 cm.

4. Construct a right triangle whose base is 7.5cm. and sum of its hypotenuse and other side is 15cm.

5. Construct a segment of a circle on a chord of length 5cm. containing the following angles.

   i. 90° ii. 45° iii. 120°

WHAT HAVE WE DISCUSSED?

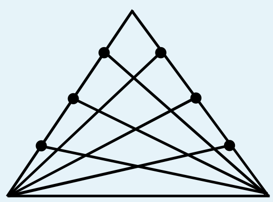
F3C7L5

1. A geometrical construction is the process of drawing geometrical figures using only two instruments - an ungraduated ruler and a compass.
2. Construction of geometrical figures of the following with justifications (Logical proofs)
   - Perpendicular bisector of a given line segment.
   - bisector of a given angle.
   - Construction of 60° angle at the initial point of a given ray.
3. To construct a triangle, given its base, a base angle and the sum of other two sides.
4. To construct a triangle given its base, a base angle and the difference of the other two sides.
5. To construct a triangle, given its perimeter and its two base angle.
6. To construct a circle segment given a chord and a chord angle.

Brain Teaser

How many triangles are threre in the figure ?

(It is a 'Cevian' write formula of a triangle - named in honour of Mathematician Ceva)



(**Hint :** Let the number of lines drawn from each vertex to the opposite side be 'n')

Chapter

14

Probability

Probability theory is nothing but common sense reduced to calculation.

- Pierre-Simon Laplace

14.1 Introduction

Siddu and Vivek are classmates. One day during their lunch they are talking to each other.

Observe their conversation

Siddu : Hello Vivek , What are you going to do in the evening today?

Vivek : Most likely, I will watch India v/s Australia cricket match.

Siddu : Whom do you think will win the toss ?

Vivek : Both teams have equal chance to win the toss. Do you watch the cricket match at home?

Siddu : There is no chance for me to watch the cricket at my home. Because my T.V. is under repair.

Vivek : Oh! then come to my home, we will watch the match together.

Siddu : I will come after doing my home work.

Vivek : Tomorrow is 2nd october. We have a holiday on the occasion of Gandhiji's birthday. So why don't you do your home work tomorrow?

Siddu : No, first I will finish the homework then I will come to your home.

Vivek : Ok.

Consider the following statements from the above conversation:

Most likely, I will watch India v/s Australia cricket match

There is ***no chance*** for me to watch the cricket match.

Both teams have ***equal chance*** to win the toss.

Here Vivek and Siddu are making judgements about the chances of the particular occurrence.

In many situations we make such statements and use our past experience and logic to take decisions. For example. It is a bright and pleasant sunny day. I need not carry my umbrella and will take a chance to go.

However, the decisions may not always favour us. Consider the situation. "Mary took her umbrella to school regularly during the rainy season. She carried the umbrella to school for many days but it did not rain during her walk to the school. However, by chance, one day she forgot to take the umbrella and it rained heavily on that day".

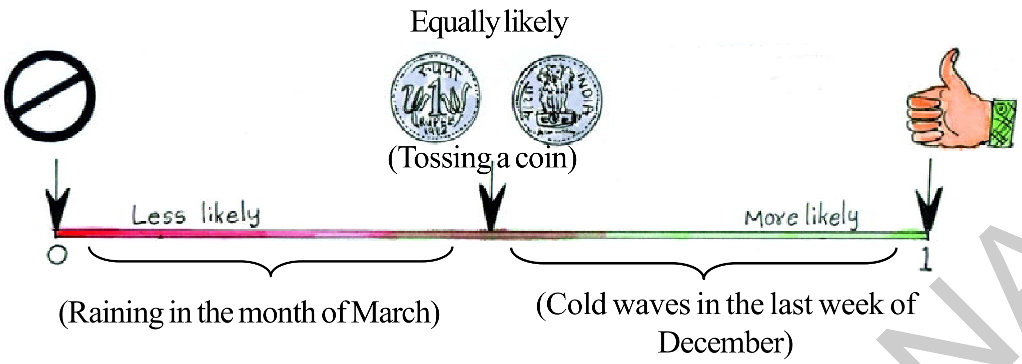
Usually the summer begins from the month of March, but one day in that month there was a heavy rainfall in the evening. Luckily Mary escaped becoming wet, because she carried umbrella on that day as she does daily.

Thus we take a decision by guessing the future happening that is whether an event occurs or not. In the above two cases, Mary guessed the occurrence and non-occurrence of the event of raining on that day. Our decision may favour us and sometimes may not. (Why?)

We try to measure numerically the chance of occurrence or non-occurrence of some events just as we measure many other things in our daily life. This kind of measurement helps us to take decision in a more systematic manner. Therefore we study probability to figure out the chance of something happening.

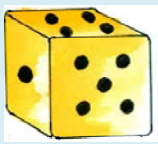
Before measuring numerically the chance of happening that we have discussed in the above situations, we grade it using the following terms given in the table. Let us observe the following table.

Term	**Chance**	**Examples from conversation**
certain	something that must occur	Gandhiji's birthday is on 2nd October.
more likely	something that would occur with great chance	Vivek watching the cricket match
equally likely	somethings that have the same chance of occurring	Both teams winning the toss.
less likely	Something that would occur with less chance	Vivek doing homework on the day of cricket match.
impossible	Something that cannot happen.	Sidhu watching the circket match at his home.

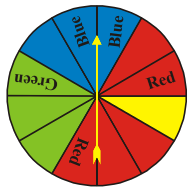


Do This

1. Observe the table given in the previous page and give some other example for each term.
2. Classify the following statements into the categories ***less likely, equally likely, more likely.***
   a) Rolling a dice* and getting a number 5 on the top face.
   b) Cold waves in your village in the month of November.
   c) India winning the next soccer(foot ball)world cup
   d) Getting a tail or head when a coin is tossed.
   e) Winning the jackpot for your lottery ticket.



14.2 Probability



spinner

14.2.1 Random experiment and outcomes

To understand and measure the chance, we perform the experiments like tossing a coin, rolling a dice and spining the spinner etc.

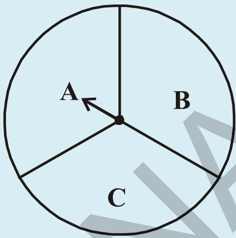
When we toss a coin we have only two possible results, head or tail. Suppose you are the captain of a cricket team and your friend is the captain of the other cricket team. You toss the coin and ask your friend to choose head or tail. Can you control the result of the toss? Can you get a head or tail that you want? In an ordinary coin that is not possible. The chance of getting either is same and you cannot say what you would get. Such an experiment known as 'random experiment'. In such experiments though we know the possible outcomes before conducting the experiment, we cannot predict the exact outcome that occurs at a particular time, in advance. The outcomes of random experiments may be equally likely or may not be. In the coin tossing experiment head or tail are two possible outcomes.

* A dice is a well balanced cube with its six faces marked with numbers from 1 to 6, one number on each face. Sometimes dots appear in place of numbers.

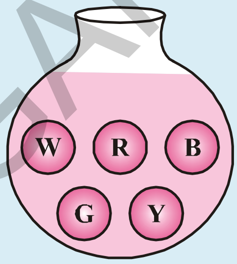
TRY THESE

1. If you try to start a scooter , What are the possible outcomes?
2. When you roll a dice, What are the six possible outcomes?
3. When you spin the wheel shown, What are the possible outcomes?

   (Out comes here means the possible sector where the pointer stops)



4. You have a jar with five identical balls of different colours (White, Red, Blue, Grey and Yellow) and you have to pickup (draw) a ball without looking at it. List the possible outcomes you get.



THINK, DISCUSS AND WRITE

In rolling a dice.

- Does the first player have a greater chance of getting a six on the top face?
- Would the player who played after him have a lesser chance of getting a six on the top face?
- Suppose the second player got a six on the top face. Does it mean that the third player would not have a chance of getting a six on the top face?

14.2.2 Equally likely outcomes

When we toss a coin or roll a dice , we assume that the coin and the dice are fair and unbiased (i.e. for each toss or roll the chance of all possibilities is equal). We conduct the experiment many times and collect the observations. Using the collected data, we find the measure of chance of occurrence of a particular happening.

A coin is tossed several times and the result is noted. Let us look at the result sheet where we keep on increasing the tosses.

Number of tosses	Tally marks (Heads)	Number of heads	Tally mark (Tails)	Number of tails
50	𝍸 𝍸 𝍸 𝍸 ||	22	𝍸 𝍸 𝍸 𝍸 𝍸 |||	28
60	𝍸 𝍸 𝍸 𝍸 𝍸 |	26	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 ||||	34
70	......	30	......	40
80	......	36	......	44
90	......	42	......	48
100	......	48	......	52

We can observe from the above table as you increase the number of tosses, the number of heads and the number of tails come closer to each other.

Do This

Toss a coin for number of times as shown in the table. And record your findings in the table.

No. of Tosses	Number of heads	No. of tails
10		
20		
30		
40		
50		

What happens if you keep on increasing the number of tosses.

This could also be done with a dice, roll it for large number of times and observe.

No. of times Die rolled	Number of times each outcome occured (i.e. each number appearing on the top face)					
	1	2	3	4	5	6
25	4	3	9	3	3	3
50	9	5	12	9	8	7
75	14	10	16	12	10	13
100	17	19	19	16	13	16
125	25	20	24	18	16	22
150	28	24	28	23	21	26
175	31	30	33	27	26	28
200	34	34	36	30	32	34
225	37	38	40	34	38	38
250	40	40	43	40	43	44
275	44	41	47	47	47	49
300	48	47	49	52	52	52

From the above table, it is evident that rolling a dice for a larger number of times, the each of six outcomes, becomes almost equal to each other.

From the above two experiments, we may say that the different outcomes of the experiment are equally likely. This means each of the outcome has equal chance of occurring.

14.2.3 Trials and Events

In the above experiments each toss of a coin or each roll of a dice is a **Trial** or **Random experiment**.

Consider a trial of rolling a dice,

How many possible outcomes are there to get a number 5 more than 5 on the top face?

It is only one (i.e. 5, 6)

How many possible outcomes are there to get an even number on the top face?

They are 3 outcomes (2, 4 and 6).

Thus each specific outcome or the collection of specific outcomes make an **Event**.

In the above trial getting a number more than 5 and getting an even number on the top face are two events. Note that event need not necessarily a single outcome. But, every outcome of a random experiment is an event.

Here we understand the basic idea of the event, more could be learnt on event in higher classes.

14.2.4 Linking the chance to Probability

Consider the experiment of tossing a coin once. What are the outcomes? There are only two outcomes Head or Tail and both outcomes are equally likely.

What is the chance of getting a head?

It is one out of two possible outcomes i.e. $\frac{1}{2}$. In other words it is expressed as the probability of getting a head when a coin is tossed is $\frac{1}{2}$, which is represented by

$$P(H) = \frac{1}{2} = 0.5 \text{ or } 50\%$$

What is the probability of getting a tail?

Now take the example of rolling a dice. What are the possible outcomes in one roll? There are six equally likely outcomes 1,2,3,4,5,or 6.

What is the probability of getting an odd number on the top face?

1, 3 or 5 are the three favourable outcomes out of six total possible outcomes. It is $\frac{3}{6}$ or $\frac{1}{2}$

We can write the formula for Probability of an event 'A'

$$P(A) = \frac{\text{Number of favourable outcomes for event 'A'}}{\text{Number of total possible outcomes}}$$

Now let us see some examples :

Example 1: If two identical coins are tossed simultaneously. Find (a) the possible outcomes, (b) the number of total outcomes, (c) the probability of getting two heads, (d) probability of getting atleast one head, (e) probability of getting no heads and (f) probability of getting only one head.

Solution : (a) The possible outcomes are

Coin 1	Coin 2
Head	Head
Head	Tail
Tail	Head
Tail	Tail

b) Number of total possible outcomes is 4

c) Probability of getting two heads

$$= \frac{\text{Number of favourable outcomes of getting two heads}}{\text{Number of total possible outcomes}} = \frac{1}{4}$$

d) Probability of getting atleast one head $= \frac{3}{4}$

[At least one head means getting a head one or more number of times]

e) Probability of getting no heads $= \frac{1}{4}$.

e) Probability of getting only one head $= \frac{2}{4} = \frac{1}{2}$.

Do This

1. If three coins are tossed simultaneously then write their outcomes.
   a) All possible outcomes
   b) Number of possible outcomes
   c) Find the probability of getting at least one head
      (getting one or more than one head)
   d) Find the Probability of getting at most two heads
      (getting Two or less than two heads)
   e) Find the Probability of getting no tails

Example 2 : (a) Write the probability of getting each number on the top face when a dice was rolled in the following table. (b) Find the sum of the probabilities of all outcomes.

Solution : (a) Out of six possibilities the number 4 occurs once hence probability is 1/6. Similarly we can fill the table for the remaining values.

Outcome	1	2	3	4	5	6
Probability (P)				1/6		

(b) The sum of all probabilities

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6)$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

We can generalize that

Sum of the probabilities of all the outcomes of a random experiment is always 1

TRY THIS

Find the probability of each event when a dice is rolled once

Event outcome(s)	Favourable outcome	Number of favourable outcome(s)	Total possible outcomes	Number possible outcomes	Probability = $\frac{\text{Number of favourable outcomes}}{\text{Number of total possible outcomes}}$
Getting a number 5 on the top face	5	1	1, 2, 3, 4, 5 and 6	6	1/6
Getting a number greater than 3 on the top face					
Getting a prime number on the top face					
Getting a number less than 5 on the top face					
Getting a number that is a factor of 6 on the top face					
Getting a number greater than 7 on the top face					
Getting a number that is a Multiple of 3 on the top face					
Getting a number 6 or less than 6 on the top face					

You can observe that

The probability of an event always lies between 0 and 1 (0 and 1 inclusive)

$$0 \le \text{probability of an event} \le 1$$

a) The probability of an event which is certain = 1

b) The probability of an event which is impossible = 0

14.2.5 Conduct your own experiments

1. We would work here in groups of 3-4 students each. Each group would take a coin of the same denomination and of the same type. In each group one student of the group would toss the coin 20 times and record the data. The data of all the groups would be placed in the table below (Examples are shown in the table).

Group No.	No. of tosses	Cumulative tosses of groups	Number of heads	Cumulative No. of heads	$\frac{\text{Cumulative heads}}{\text{total times tossed}}$	$\frac{\text{Cumulative tails}}{\text{total times tossed}}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	20	20	7	7	$\frac{7}{20}$	$\frac{20-7}{20} = \frac{13}{20}$
2	20	40	14	21	$\frac{21}{40}$	$\frac{40-21}{40} = \frac{19}{40}$
3	20	60				
4	20	80				
5	20	100				
6	.....	....				
7	....	....				

What happens to the value of the fractions in (6) and (7) when the total number of tosses of the coin increases? Could you see that the values are moving close to the probability of getting a head and tail respectively.

2. In this activity also we would work in groups of 3-4. One student from each group would roll a dice for 30 times. Other students would record the data in the following table. All the groups should have the same kind of dice so that all the throws will be treated as the throws of the same dice.

No. of times Dice rolled	Number of times the following outcomes turn up					
	1	2	3	4	5	6
30						

Complete the following table, using the data obtained from all the groups :

Group(s)	Number of times 1 turned up	Total number of times a dice is rolled	$\dfrac{\text{Number of times 1 turned up}}{\text{Total number of times a dice is rolled}}$
(1)	(2)	(3)	(4)
1^{st}			
$1^{st}+2^{nd}$			
$1^{st}+2^{nd}+3^{rd}$			
$1^{st}+2^{nd}+3^{rd}+4^{th}$			
$1^{st}+2^{nd}+3^{rd}+4^{th}+5^{th}$			

What do you observe as the number of rolls increases; the fractions in cloumn (4) move closer to $\frac{1}{6}$? We performed the above experiment for the outcome 1. Check the same for the outcome 2 and the outcome 5.

What can you conclude about the values you get in column (4) and compare these with the probabilities of getting 1, 2, and 5 on rolling a dice?

3. What would happen, if we toss two coins simultaneously? We could have either both coins showing head, both showing tail or one showing head and one showing tail. Would the possibility of occurrence of these three be the same? Think about this while you do this group activity.

Divide class into small groups of 4 each. Let each group take two coins. Note that all the coins used in the class should be of the same denomination and of the same type. Each group would throw the two coins simultaneously 20 times and record the observations in a table.

No. of times two coins tossed	No. of times no head turns up	Number of times one head turns up	Number of times two heads turns up
20			

All the groups should now make a cummulative table:

Group(s)	Number of times two coins are tossed	Number of times no head turns up	Number of times one head turns up	Number of times two heads turns up
1^{st}				
$1^{st} + 2^{nd}$				
$1^{st} + 2^{nd} + 3^{rd}$				
$1^{st} + 2^{nd} + 3^{rd} + 4^{th}$				
....				

Now we find the ratio of the number of times no head turns up to the total number of times two coins are tossed. Do the same for the remaining events.

Fill the following table:

Group(s)	$\dfrac{\text{No. of times no head}}{\text{Total tosses}}$	$\dfrac{\text{No. of times one head}}{\text{Total tosses}}$	$\dfrac{\text{No. of times two heads}}{\text{Total tosses}}$
(1)	(2)	(3)	(4)
Group 1^{st}			
Group $1 + 2^{nd}$			
Group $1 + 2 + 3^{rd}$			
Group $1 + 2 + 3 + 4^{th}$			
....			

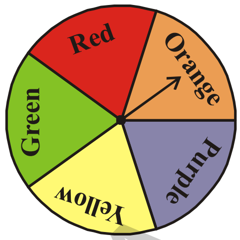
As the number of tosses increases, the values of the columns (2), (3) and (4) get closer to 0.25, 0.5 and 0.25 respectively.

Example-3: A spinner was spun 1000 times and the frequency of outcomes was recorded as in given table:

Out come	Red	Orange	Purple	Yellow	Green
Frequency	185	195	210	206	204

Find (a) How many possible outcomes can you see in the spinner? What are they? (b) Compute the probability of each colour. (c) Find the ratio of each colour to the total number of times that the spinner spun (use the table)

Solution :



spinner

(a) The possible outcomes are 5. They are red, orange, purple, yellow and green. Here all the five colours occupy equal areas in the spinner. So, they are all equally likely.

(b) Let us compute the probability of each event.

$$P(\text{Red}) = \frac{\text{Favourable outcomes of red}}{\text{Total number of possible outcomes}}$$

$$= \frac{1}{5} = 0.2.$$

Similarly

P(Orange), P(Purple), P(Yellow) and P(Green) is also $\frac{1}{5}$ or 0.2.

(c) From the experiment the frequency was recorded in the table

$$P(\text{Red}) = \text{Ratio for red} = \frac{\text{No. of outcomes of red in the above experiment}}{\text{Number of times the spinner was spun}}$$

$$= \frac{185}{1000} = 0.185$$

Similarly, we can find the corresponding ratios for orange, purple, yellow and green are 0.195, 0.210, 0.206 and 0. 204 respectively.

Can you see that each of the ratio is approximately equal to the probability which we have obtained in (b) [i.e. before conducting the experiment]

Example-4. The following table gives the ages of audicence in a theatre. Each person was given a serial number and a person was selected randomly for the bumper prize by choosing a serial number. Find the probability of each event.

Age	Male	Female
Under 2	3	5
3 - 10 years	24	35
11 - 16 years	42	53
17 - 40 years	121	97
41- 60 years	51	43
Over 60	18	13

Total number of audience : 505

a) The probability of audience of age less than or equal to 10 years

Solution : The audience of age less than or equal to 10 years = 24 + 35 + 5 + 3 = 67

Total number of people = 505

$$P(\text{audicence of age} \leq 10 \text{ years}) = \frac{67}{505}$$

b) The probability of female audience of age 16 years or younger

Solution : The female audience with age less than or equal 16 years = 53 + 35 + 5 = 93

$P(\text{female audicence of age} \leq 16 \text{ years}) = 93/505$

c) The probability of male audience of age 17 years or above

Solution : The male audience of age 17 years or above = 121 + 51 + 18 = 190

$$P(\text{male audience of age} \geq 17 \text{ years}) = \frac{190}{505} = \frac{38}{101}$$

d) The probability of audience of age above 40 years

Solution : The audience of age above 40 years = 51+43+18+ 13 = 125

$$P(\text{audience of age} > 40 \text{ years}) = \frac{125}{505} = \frac{25}{101}$$

e) The probability of the person watching the movie is not a male

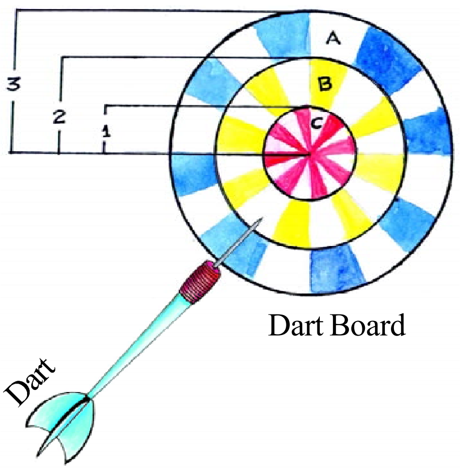
Solution : The number of persons watching the movie is not a male

= 5 + 35 + 53 + 97 + 43 + 13 = 246

$$P(\text{A person watching movie is not a male}) = \frac{246}{505}$$

Example-5 : Assume that a dart will hit the dart board and each point on the dart board is equally likely to be hit in all the three concentric circles where radii of concetric circles are 3 cm, 2 cm and 1 cm as shown in the figure below.

Find the probability of a dart hitting the board in the region A. (The outer ring)



Dart Board

Solution : Here the event is hitting in region A.

The Total area of the circular region with radius 3 cm

$= \pi(3)^2$

Area of circular region A (i.e. ring A) = $\pi(3)^2 - \pi(2)^2$

Probability of the dart hitting the board in region A is P(A)

$$P(A) = \frac{\text{Area of circular region A}}{\text{Total Area of the circular region}}$$

$$= \frac{\pi(3)^2 - \pi(2)^2}{\pi(3)^2}$$

$$= \frac{9\pi - 4\pi}{9\pi} = \frac{5}{9}$$

Remember

Area of a circle = πr^2

Area of a ring = $\pi R^2 - \pi r^2$

Try These

From the figure given in example 5.

1. Find the probability of the dart hitting the board in the circular region B (i.e. ring B).
2. Without calculating, write the percentage of probability of the dart hitting the board in circular region C (i.e. ring C).

14.3 Uses of Probability in Real Life

- Meteorological department predicts the weather by observing trends from the data collected over many years in the past.
- Insurance companies calculate the probability of happening of an accident or casuality to determine insurance premiums.
- "An exit poll" is taken after the election . It is surveying the people to which party they have voted. This gives an idea of winning chances of each candidate and predictions are made accordingly.

EXERCISE - 14.1

1. A dice has six faces numbered from 1 to 6. It is rolled and the number on the top face is noted. When this is treated as a random trial.

   a) What are the possible outcomes ?

   b) Are they equally likely? Why?

   c) Find the probability of a composite number turning up on the top face.

2. A coin is tossed 100 times and the following outcomes are recorded

   Head:45 times   Tails:55 times from the experiment

   a) Compute the probability of each outcomes.

   b) Find the sum of probabilities of all outcomes.

3. A spinner has four colours as shown in the figure. When we spin it once, find

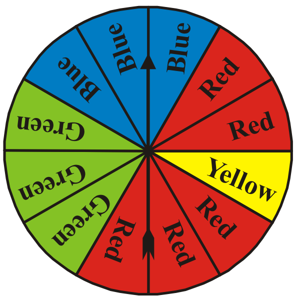
   a) At which colour, is the pointer more likely to stop?

   b) At which colour, is the pointer less likely to stop?

   c) At which colours, is the pointer equally likely to stop?

   d) What is the chance the pointer will stop on white?

   e) Is there any colour at which the pointer certainly stops?



4. A bag contains five green marbles, three blue marbles, two red marbles, and two yellow marbles. One marble is drawn out randomly.

   a) Are the four different colour outcomes equally likely? Explain.

   b) Find the probability of drawing each colour marble

   i.e. , P(green), P(blue), P(red) and P(yellow)

   c) Find the sum of their probabilities.

5. A letter is chosen from English alphabet. Find the probability of the letters being

   a) A vowel   b) a letter that comes after P

   c) A vowel or a consonant   d) Not a vowel

6. Eleven bags of wheat flour, each marked 5 kg, actually contained the following weights of flour (in kg):

4.97, 5.05, 5.08, 5.03, 5.00, 5.06, 5.08, 4.98, 5.04, 5.07, 5.00

Find the probability that any of these bags chosen at random contains more than 5 kg of flour.

7. An insurance company selected 2000 drivers at random (i.e., without any preference of one driver over another) in a particular city to find a relationship between age and accidents. The data obtained is given in the following table:

Age of Drivers (in years)	**Accidents in one year**				**More than 3 accidents**
	0	**1**	**2**	**3**	
18-29	440	160	110	61	35
30- 50	505	125	60	22	18
Over 50	360	45	35	15	9

Find the probabilities of the following events for a driver chosen at random from the city:

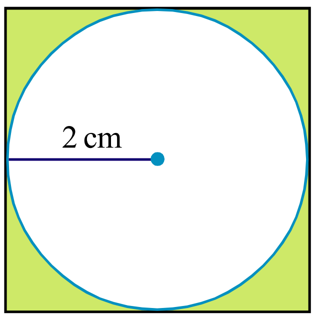
(i) The driver being in the age group 18-29 years and having exactly 3 accidents in one year.

(ii) The driver being in the age group of 30-50 years and having one or more accidents in a year.

(iii) Having no accidents in the year.

8. What is the probability that a randomly thrown dart hits the square board in shaded region?

(Take $\pi = \frac{22}{7}$ and express answer in percentage)



WHAT HAVE WE DISCUSSED?

E6Z3A9

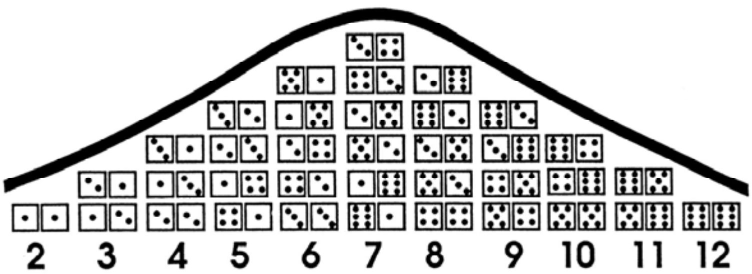
- There is use of words like most likely, no chance, equally likely in daily life, are showing the manner of chance and judgement.
- There are certain experiments whose outcomes have equal chance of occurring. Outcomes of such experiments are known as **equally likely** outcomes.
- An event is a collection of a specific outcome or some of the specific outcomes of the experiment.
- In some random experiments all outcomes have equal chance of occurring.
- As the number of trials increases, the probability of all equally likely outcomes come very close to each other.
- The probability of an event A

$$P(A) = \frac{\text{Number of favourable outcomes of event A}}{\text{Number of total possible outcomes}}$$

- The probability of an event which is certain = 1.
- The probability of an event which is impossible = 0
- The probability of an event always lies between 0 and 1 (0 and 1 inclusive).

Do you Know?

The diagram below shows the 36 possible outcomes when a pair of dice are thrown. It is interesting to notice how the frequency of the outcomes of different possible numbers (2 to 12).



This curve illustrate the Gaussian curve, name after 19th century famous mathematician Carl Friedrich Gauss.

Chapter

15

Proofs in Mathematics

15.1 Introduction

We come across many statements in our daily life. We gauge the worth of each statement. Some statements we consider to be appropriate and true and some we dismiss. There are some we are not sure of. How do we make these judgements? In case there is a statement of conflict about loans or debts. You want to claim that bank owes your money then you need to present documents as evidence of the monetary transaction. Without that, people would not believe you. If we think carefully we can see that in our daily life we need to prove if a statement is true or false. In our conversations in daily life we sometimes do not consider to prove or check statements and accept them without serious examination. That however will not be accepted in mathematics. Consider the following:

1. The sun rises in the east.
2. $3 + 2 = 5$
3. New York is the capital of USA.
4. $4 > 8$
5. How many siblings do you have?
6. Goa has better football team than Bengal.
7. Rectangle has 4 lines of symmetry.
8. $x + 2 = 7$
9. Please come in.
10. The probability of getting two consecutive 6's on throws of a 6 sided dice.
11. How are you?
12. The sun is not stationary but moving at high speed all the time.
13. $x < y$
14. Where do you live?

In the above sentences you find some sentences are false. For example, 4>8, and New York is not the capital of USA. You find some sentences are correct.

These include "sun rises in the east." The probability of getting two consecutive 6's, the Sun is not stationary etc.

Besides those there are some other sentences that are true for some cases but not true for other cases, for example $x + 2 = 7$ is true only when $x = 5$ and $x < y$ is only true for those values of x and y where x is less than y.

Look at the other sentences which of them are clearly false or clearly true. Such type of sentences are called statements. We say these statements that can be judged on some criteria, no matter by what process for their being true or false.

Think about these:

1. Please ignore this notice.....
2. The statement I am making is false.
3. This sentence has some words.
4. You may find water on the moon.

Can you say whether these sentences are true or false? Is there any way to check them being true or false?

Look at the first sentence, if you ignore the notice, you do that because it tells you to do so. If you do not ignore the notice, then you have paid some attention to it. So you can never follow it and being an instruction it cannot be judged on a true/false scale. 2^{nd} and 3^{rd} sentences are talking about themselves. 4th sentence have words that show only likely or possibility and hence ambiguity of being on both sides.

The sentences which are talking about themselves and the sentences with possibility are not statements.

Do This

Make 5 more sentences and check whether they are statements or not. Give reasons.

15.2 Mathematical Statements

We can write infinately large number of sentences. You can think the kind of sentences you use and can you count the number of sentences you speak? Not all these however, they can be judged on the criteria of false and true. For example, consider, please come in. Where do you live? Such sentences can also be very large in number.

All these the sentences are not statements. Only those that can be judged either true or false but not both are statements. The same is true for mathematical statements. A mathemtical statement can not be ambiguous. In mathematics a statement is only acceptable if it is either true or false. Consider the following sentences:

1. 3 is a prime number.
2. Product of two odd integers is even.
3. For any real number x; $4x + x = 5x$
4. The earth has one moon.
5. Ramu is a good driver.
6. Bhaskara has written a book "Leelavathi".
7. All even numbers are composite.
8. A rhombus is a square.
9. $x > 7$.
10. 4 and 5 are relative primes.

11. Silver fish is made of silver.

12. Humans are meant to rule the earth.

13. For any real number x, $2x > x$.

14. Havana is the capital of Cuba.

Which of these are mathematical and which are not mathematical statements?

15.3 Verifying the Statements

Let us consider some of the above sentences and discuss them as follows:

Example-1. We can show that (1) is true from the definition of a prime number.

Which of the sentences from the above list are of this kind of statements that we can prove mathematically? (Try to prove).

Example-2. "Product of two odd integers is even". Consider 3 and 5 as the odd integers. Their product is 15, which is not even.

Thus it is a statement which is false. So with one example we have showed this. Here we are able to verify the statement using an example that runs counter to the statement. Such an example, that counters a statement is called a counter example.

Try This

Which of the above statements can be tested by giving a counter example ?

Example-3. Among the sentences there are some like **"Humans are meant to rule the earth"** or **"Ramu is a good driver**."

These sentences are ambiguous sentences as the meaning of ruling the earth is not specific. Similarly, the definition of a good driver is not specified.

We therefore recognize that a 'mathematical statement' must comprise of terms that are understood in the same way by everyone.

Example-4. Consider some of the other sentences like

The earth has one Moon.

Bhaskara has written the book "Leelavathi"

Think about how would you verify these to consider as statements?

These are not ambiguous statements but needs to be tested. They require some observations or evidences. Besides, checking this statement cannot be based on using previously known results. The first sentence require observations of the solar system and more closely of the earth. The second sentence require other documents, references or some other records.

Mathematical statements are of a distinct nature from these. They cannot be proved or justified by getting evidence while as we have seen, they can be disproved by finding an example counter to the statement.

In the statement for any real number $2x > x$, we can take $x = -1$ or $-\frac{1}{2}$ .... and disprove the statement by giving counter example. You might have also noticed that $2x > x$ is true with a condition on x i.e. x belong to set N.

Example-5. Restate the following statements with appropriate conditions, so that they become true statements.

i. For every real number x, $3x > x$.

ii. For every real number x, $x^2 \geq x$.

iii. If you divide a number by two, you will always get half of that number.

iv. The angle subtended by a chord of a circle at a point on the circle is 90°.

v. If a quadrilateral has all its sides equal, then it is a square.

Solution :

i. If $x > 0$, then $3x > x$.

ii. If $x \leq 0$ or $x \geq 1$, then $x^2 \geq x$.

iii. If you divide a number other than 0 by 2, then you will always get half of that number.

iv. The angle subtended by a diameter of a circle at a point on the circle is 90°.

v. If a quadrilateral has all its sides and interior angles equal, then it is a square.

Exercise - 15.1

1. State whether the following sentences are always true, always false or ambiguous. Justify your answer.
   i. There are 27 days in a month.
   ii. Makarasankranthi falls on Friday.
   iii. The temperature in Hyderabad is 2°C.
   iv. The earth is the only planet where life exist.
   v. Dogs can fly.
   vi. February has only 28 days.

2. State whether the following statements are true or false. Give reasons for your answers.
   i. The sum of the interior angles of a quadrilateral is 350°.
   ii. For any real number x, $x^2 \geq 0$.
   iii. A rhombus is a parallelogram.
   iv. The sum of two even numbers is even.
   v. Square numbers can be written as the sum of two odd numbers.

3. Restate the following statements with appropriate conditions, so that they become true statements.
   i. All numbers can be represented as the product of prime factors.
   ii. Two times a real number is always even.
   iii. For any x, $3x + 1 > 4$.
   iv. For any x, $x^3 \geq 0$.
   v. In every triangle, a median is also an angle bisector.

4. Disprove, by finding a suitable counter example, the statement $x^2 > y^2$ for all $x > y$.

15.4 REASONING IN MATHEMATICS

We human beings are naturally curious. This curiosity makes us to interact with the world. What happens if we push this? What happens if we stuck our finger in that? What happens if we make various gestures and expressions? From this experimentation, we begin to form a more or less consistant picture of the way that the physical world behaves. Gradually, in all situations, we make a shift from

'What happens if.....?' to *'this will happen if'*

The experimentation moves on to the exploration of new ideas and the refinement of our world view of previously understood situations. This description of the playtime pattern very nicely models the concept of 'making and testing hypothesis.' It follows this pattern:

- Make some observations, Collect data based on the observations.
- Draw conclusion (called a 'hypothesis') which will explain the pattern of the observations.
- Test out hypothesis by making some more targeted observations.

So, we have

- A **hypothesis** is a statement or idea which gives an explanation to a series of observations.

Sometimes, following observation, a hypothesis will clearly need to be refined or rejected. This happens if a *single* contradictory observation occurs. In general we use word conjecture in mathematics instead of hypothesis. You will learn the similarities and difference between these two in the higher classes.

15.4.1 Using deductive reasoning in hypothesis testing

There is often confusion between the ideas surrounding proof, making and testing an experimental hypothesis which is mathematics, which is science. The difference is rather simple:

- Mathematics is based on *deductive reasoning* : a proof is a logical deduction from a set of clear inputs.
- Science is based on *inductive reasoning* : hypotheses are strengthened or rejected based on an accumulation of experimental evidence.

Of course, to be good at science, you need to be good at deductive reasoning, although experts at deductive reasoning need not be mathematicians.

Detectives, such as Sherlock Holmes and Hercule Poirot, are such experts : they collect evidence from a crime scene and then draw logical conclusions from the evidence to support the hypothesis that, for example, person M. committed the crime. They use this evidence to create sufficiently compelling deductions to support their hypothesis *beyond reasonable doubt*. The key word here is 'reasonable'.

15.4.2 Deductive Reasoning

The main logical tool used in establishing the truth of an **unambiguous** statement is *deductive reasoning.* To understand what deductive reasoning is all about, let us begin with a puzzle for you to solve.

You are given four cards. Each card has a number printed on one side and a letter on the other side.

$$\boxed{A} \quad \boxed{V} \quad \boxed{8} \quad \boxed{5}$$

Suppose you are told that these cards follow the rule:

"If a card has an odd number on one side, then it has a vowel on the other side."

What is the **smallest number** of cards you need to turn over to check if the rule is true?

Of course, you have the option of turning over all the cards and checking. But can you manage with turning over a fewer number of cards?

Notice that the statement mentions that a card with an odd number on one side has a vowel on the other. It does not state that a card with a vowel on one side must have an odd number on the other side. That may or may not be so. The rule also does not state that a card with an even number on one side must have a consonant on the other side. It may or may not.

So, do we need to turn over $\boxed{A}$? No! Whether there is an even number or an odd number on the other side, the rule still holds.

What about $\boxed{8}$? Again we do not need to turn it over, because whether there is a vowel or a consonant on the other side, the rule still holds.

But you do need to turn over $\boxed{V}$ and 5. if $\boxed{V}$ has an odd number on the other side, then the rule has been broken. Similarly, if $\boxed{5}$ has a consonant on the other side, then the rule has been broken.

The kind of reasoning we have used to solve the puzzle is called **deductive reasoning**. It is called 'deductive' because we arrive at (i.e., deduce or infer) a result or a statement from a previously established statement using logic. For example, in the puzzle by a series of logical arguments we deduced that we need to turn over only $\boxed{V}$ and $\boxed{5}$.

Deductive reasoning also helps us to conclude that a particular statement is true, because it is a special case of a more general statement that is known to be true. For example, once we prove that the product of two even numbers is always even, we can immediately conclude (without computation) that 56702 × 19992 is even simply because 56702 and 19992 are even.

Consider some other examples of deductive reasoning:

i. If a number ends in '0' it is divisible by 5. 30 ends in 0.

From the above two statements we can deduce that 30 is divisible by 5 because it is given that the number ends in 0 is divisible by 5.

ii. Some singers are poets. All lyricists are Poets.

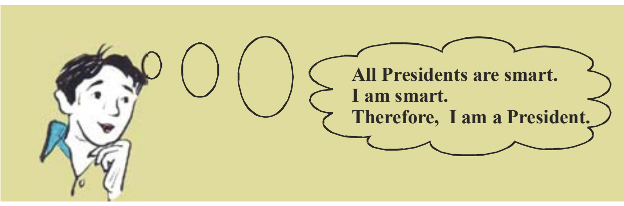
Here the deduction based on two statemens is wrong. (Why?) All lyricist are poets (wrong). Because we are not sure about it. There are three posibilities (i) all lyricists could be poets, (ii) few could be poets or (iii) none of the lyricists is a poet.

You may come to a conclusion that if - then conditional statement comes into deductive reasoning. In mathematics we use this reasoning a lot like if linear pair of angles are 180°. Then only the sum of angles in a triangle is equal to 180°. Like wise if we are using decimal number system to write a number 5. If we use the binary system we represent the quantity by 101.

Unfortunately we do not always use correct reasoning in our daily life. We often come to many conclusions based on faulty reasoning. For example, if your friend does not talk to you one day, then you may conclude that she is angry with you. While it may be true that "if she is angry at me she will not talk to me", it may also be true that "if she is busy, she will not talk to me. Why don't you examine some conclusions that you have arrived at in your day-to-day existence, and see if they are based on valid or faulty reasoning?

Exercise - 15.2

1. Use deductive reasoning to answer the following:
   i. Human beings are mortal. Jeevan is a human being. Based on these two statements, what can you conclude about Jeevan ?
   ii. All Telugu people are Indians. X is an Indian. Can you conclude that X belongs to Telugu people.
   iii. Martians have red tongues. Gulag is a Martian. Based on these two statements, what can you conclude about Gulag?
   iv. What is the fallacy in the Raju's reasoning in the cartoon below?



2. Once again you are given four cards. Each card has a number printed on one side and a letter on the other side. Which are the only two cards you need to turn over to check whether the following rule holds?

"If a card has a consonant on one side, then it has an odd number on the other side."

B	3	U	8

3. Think of this puzzle What do you need to find a chosen number from this square?

Four of the clues below are true but do nothing to help in finding the number.

Four of the clues are necessary for finding it.

Here are eight clues to use:

a. The number is greater than 9.

b. The number is not a multiple of 10.

c. The number is a multiple of 7.

d. The number is odd.

e. The number is not a multiple of 11.

f. The number is less than 200.

g. Its ones digit is larger than its tens digit.

h. Its tens digit is odd.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

What is the number?

Can you sort out the four clues that help and the four clues that do not help in finding it? First follow the clues and strike off the number which comes out from it. Like - from the first clue we come to know that the number is not from 1 to 9 strike off them.

After completing the puzzle, see which clue is important and which is not?

15.5 Theorems, Conjectures and Axioms

So far we have discussed statements and how to check their validity. In this section, you will study how to distinguish between the three different kinds of statements, Mathematics is built up from, namely, a theorem, a conjecture and an axiom.

You have already come across many theorems before. So, what is a theorem? A mathematical statement whose truth has been established (proved) is called a *theorem*. For example, the following statements are theorems.

Theorem-15.1 : The sum of the interior angles of a triangle is 180°.

Theorem-15.2 : The product of two odd natural numbers is odd.

Theorem-15.3 : The product of any two consecutive even natural numbers is divisible by 4.

A *conjecture* is a statement which we believe as true, based on our mathematical understanding and experience, i.e., our mathematical intuition. The conjecture may turn out to be true or false. If we can prove it, then it becomes a theorem. Mathematicians often come up with conjectures by looking for patterns and making intelligent mathematical guesses. Let us look at some patterns and see what kind of intelligent guesses we can make.

While studying some cube numbers Raju noticed that "if you take three consecutive whole numbers and multiply them together and then add the middle number of the three, you get the middle number cubed"; *e.g.*, 3, 4, 5, gives $3 \times 4 \times 5 + 4 = 64$, which is a perfect cube. Does this always work? Take some more consecutive numbers and check it.

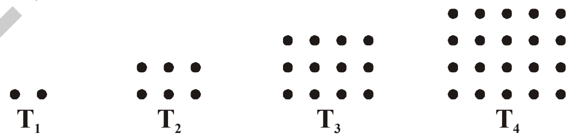
Rafi took 6, 7, 8 and checked this conjecture. Here 7 is the middle term so according to the rule $6 \times 7 \times 8 + 7 = 343$, which is also a perfect cube. Try to generalize it by taking numbers as n, $n + 1$, $n + 2$. See other example:

Example-6. The following geometric arrays suggest a sequence of numbers.

(a) Find the next three terms.

(b) Find the 100^{th} term.

(c) Find the n^{th} term.



The dots here arranged in such a way that they form a rectangle. Here $T_1 = 2$, $T_2 = 6$, $T_3 = 12$, $T_4 = 20$ and so on. Can you guess what T_5 is? What about T_6? What about T_n?

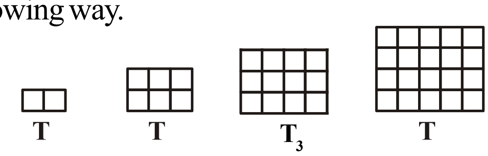
Make a conjecture about T_n.

It might help if you redraw them in the following way.

Solution :

T_1	T_2	T_3	T_4	T_5	T
2	6	12	20	?	.....

$$2 \underbrace{\quad}_{+4} 6 \underbrace{\quad}_{+6} 12 \underbrace{\quad}_{+8} 20 \underbrace{\quad}_{+10} ?$$



So, $T_5 = T_4 + 10 = 20 + 10 = 30 = 5 \times 6$

$T_6 = T_5 + 12 = 30 + 12 = 42 = 6 \times 7$ ..... Try for T_7?

$T_{100} = 100 \times 101 = 10,100$

$T_n = n \times (n + 1) = n^2 + n$

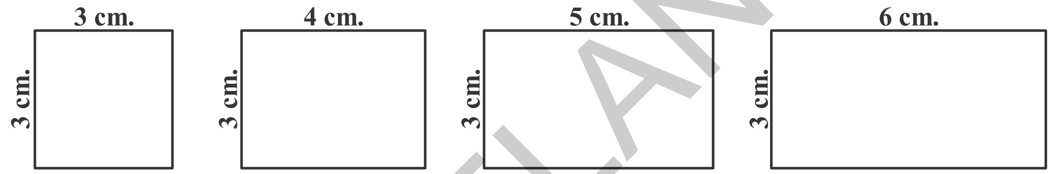
This type of reasoning which is based on examining a variety of cases or sets of data, discovering patterns and forming conclusions is called ***inductive reasoning***. Inductive reasoning is very helpful technique for making conjecture.

Gold bach the famous mathematician in 1743, put forward a pattern observe it :

$6 = 3 + 3$	$8 = 3 + 5$	$10 = 3 + 7$
$12 = 5 + 7$	$14 = 11 + 3$	$16 = 13 + 3 = 11 + 5$

From the above we observe that every even number greater than 4 can be written as the sum of two primes (not necessarily distinct primes). His conjecture has not been proved to be true or false so far. Perhaps you will prove that this result is true or false and will become a theorem.

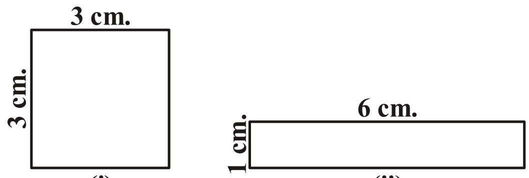
But just by looking few patterns some time lead us to a wrong conjecture like: in class 8th Janvi and Kartik while studying Area and Perimeter chapter..... observed a pattern



	(i)	(ii)	(iii)	(iv)
Perimeter :	12 cm.	14 cm.	16 cm.	18 cm.
Area :	9 cm^2	12 cm^2	15 cm^2	18 cm^2

and stated a conjecture that when the perimeter of the rectangle increases the area will also increase. What do you think? Are they right? While working on this pattern.

Inder drew some rectangles and disproved the conjecture stated by Janvi and Kartik.

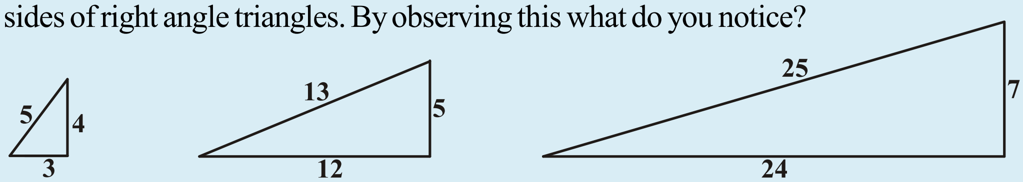


	(i)	(ii)
Perimeter :	12 cm.	14 cm.
Area :	9 cm^2	6 cm^2

We understand that while making a conjecture we have to look at all the possibilities.

Try This

Envied by the popularity of Pythagoras his disciple claimed a different relation between the sides of right angle triangles. By observing this what do you notice?



(i) (ii) (iii)

Liethagoras Theorem : In any right angle triangle the square of the smallest side equals the sum of the other sides.

Check this conjucture, whether it is right or wrong.

You might have wondered - do we need to prove every thing we encounter in mathematics and if not, why not?

In mathematics some statements are assumed to be true and are not proved, these are self-evident truths' which we take to be true without proof. These statements are called *axioms*. In chapter 3, you would have studied the axioms and postulates of Euclid. (We do not distinguish between axioms and postulates these days generally we use word postulate in geometry).

For example, the first postulate of Euclid states:

A straight line may be drawn from any point to any other point.

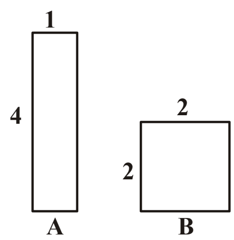
And the third postulate states:

A circle may be drawn with any centre and any radius.

These statements appear to be perfectly true and Euclid assumed them to be true. Why? This is because we cannot prove everything and we need to start somewhere, we need some statements which we accept as true and then we can build up our knowledge using the rules of logic based on these axioms.

You might then wonder why don't we just accept all statements to be true when they appear self evident. There are many reasons for this. Very often our intuition can be wrong, pictures or patterns can deceive and the only way to be sure that something is true is to prove it. For example, many of us believe that if a number is added to another number, the result will be large than the numbers. But we know that this is not always true : for example 5 + (-5) = 0, which is smaller than 5.

Also, look at the figures. Which has bigger area ?



It turns out that both are of exactly the same area, even though B appears bigger.

You might then wonder, about the validity of axioms. Axioms have been chosen based on our intuition and what appears to be self-evident. Therefore, we expect them to be true. However, it is possible that later on we discover that a particular axiom is not true. What is a safeguard against this possibility? We take the following steps:

i. Keep the axioms to the bare minimum. For instance, based only on axioms and five postulates of Euclid, we can derive hundreds of theorems.

ii. Make sure that the axioms are consistent.

We say a collection of axioms is *inconsistent*, if we can use one axiom to show that another axiom is not true. For example, consider the following two statements. We will show that they are inconsistent.

Statement-1 : No whole number is equal to its successor.

Statement-2 : A whole number divided by zero is a whole number.

(Remember, **division by zero is not defined**. But just for the moment, we assume that it is possible, and see what happens.)

From Statement-2, we get $\frac{1}{0} = a$, where a is some whole number. This implies that, 1=0. But this disproves Statement-1, which states that no whole number is equal to its successor.

iii. A false axiom will, sooner or later, result into contradiction. We say that *there is a contradiction, when we find a statement such that, both the statement and its negation are true.* For example, consider Statement-1 and Statement-2 above once again.

From Statement-1, we can derive the result that $2 \neq 1$.

Let $x = y$

$x \times x = xy$

$x^2 = xy$

$x^2 - y^2 = xy - y^2$

$(x+y)(x-y) = y(x-y)$ From Statement-2, we can cancel $(x - y)$ from both the sides.

$x + y = y$

But $x = y$

so $x + x = x$

or $2x = x$

$2 = 1$

So we have both the statements $2 = 1$ and its negation, $2 \neq 1$ are true. This is a contradiction. The contradiction arose because of the false axiom, that a whole number divided by zero is a whole number.

So, the statement we choose as axioms require a lot of thought and insight. We must make sure they do not lead to inconsistencies or logical contradictions. Moreover, the choice of axioms themselves, sometimes leads us to new discoveries.

We end the section by recalling the differences between an axiom, a theorem and a conjecture. An **axiom** is a mathematical statement which is true without proof; a **conjecture** is a mathematical statement whose truth or falsity is yet to be established; and a **theorem** is a mathematical statement whose truth has been logically established.

Exercise - 15.3

1. (i) Take any three consecutive odd numbers and find their product;

   for example, $1 \times 3 \times 5 = 15$, $3 \times 5 \times 7 = 105$, $5 \times 7 \times 9$ - .....

   (ii) Take any three consecutive even numbers and add them, say,

   $2 + 4 + 6 = 12$, $4 + 6 + 8 = 18$, $6 + 8 + 10 = 24$, $8 + 10 + 12 = 30$ and so on.

   Is there any pattern can you guess in these sums? What can you conjecture about them?

2. Observe the Pascal's triangle.

   Line-1 : $1 = 11^0$

   Line-2 : $11 = 11^1$

   Line-3 : $121 = 11^2$

   Make a conjecture about Line-4 and Line-5.

   Does your conjecture hold? Does your conjecture hold for Line-6 too?

   1
   1 1
   1 2 1
   1 3 3 1
   1 4 6 4 1

3. Look at the following pattern:

   i) $28 = 2^2 \times 7^1$, Total number of factors $(2+1)(1+1) = 3 \times 2 = 6$
   28 is divisible by 6 factors i.e. 1, 2, 4, 7, 14, 28

   ii) $30 = 2^1 \times 3^1 \times 5^1$, Total number of factors $(1+1)(1+1)(1+1) = 2 \times 2 \times 2 = 8$
   30 is divisible by 8 factors i.e. 1, 2, 3, 5, 6, 10, 15, 30

   Find the pattern.
   (**Hint :** Product of every prime base exponent +1)

4. Look at the following pattern:

   $1^2 = 1$

   $11^2 = 121$

   $111^2 = 12321$

   $1111^2 = 1234321$

   $11111^2 = 123454321$

Make a conjecture about each of the following:

$111111^2 =$

$1111111^2 =$

Check if your conjecture is true.

5. List five axioms (postulates) used in this book.

6. In a polynomial $p(x) = x^2 + x + 41$ put different values of x and find $p(x)$. Can you conclude after putting different values of x that $p(x)$ is prime for all. Is x an element of N? Put $x = 41$ in $p(x)$. Now what do you find?

15.6 What is a Mathematical Proof?

Before you study proofs in mathematics, you are mainly asked to verify statements.

For example, you might have been asked to verify with examples that "the product of two odd numbers is odd". So you might have picked up two random odd numbers, say 15 and 2005 and checked that $15 \times 2005 = 30075$ is odd. You might have done so for many more examples.

Also, you might have been asked as an activity to draw several triangles in the class and compute the sum of their interior angles. Apart from errors due to measurement, you would have found that the interior angles of a triangle add up to 180°.

What is the flaw in this method? There are several problems with the process of verification. While it may help you to make a statement you believe is true, you cannot be *sure* that it is true in *all* cases. For example, the multiplication of several pairs of even numbers may lead us to guess that the product of two even numbers is even. However, it does not ensure that the product of all pairs of even numbers is even. You cannot physically check the products of all possible pairs of even numbers because they are endless. Similarly, there may be some triangles which you have not yet drawn whose interior angles do not add up to 180°.

Moreover, verification can often be misleading. For example, we might be tempted to conclude from Pascal's triangle (Q.2 of Exercise), based on earlier verification, that $11^5 = 15101051$. But in fact $11^5 = 161051$.

So, you need another approach that does not depend upon verification for some cases only. There is another approach, namely 'proving a statement'. A process which can establish the truth of a mathematical statement based purely on logical arguments is called a *mathematical proof*.

To make a mathematical statement false, we just have to produce a single counter-example. So while it is not enough to establish the validity of a mathematical statement by checking or verifying it for thousands of cases, it is enough to produce one counter example to *disprove* a statement.

Let us look what should be our procedure to prove.

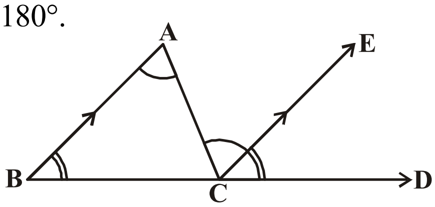
i. First we must understand clearly, what is required to prove, then we should have a rough idea how to proceed.

ii. A proof is made up of a successive sequence of mathematical statements. Each statement is a proof logically deduced from a previous statement in the proof or from a theorem proved earlier or an axiom or our hypothesis and what is given.

iii. The conclusion of a sequence of mathematically true statements laid out in a logically correct order should be what we wanted to prove, that is, what the theorem claims.

To understand that, we will analyse the theorem and its proof. You have already studied this theorem in chapter-4. We often resort to diagrams to help us to prove theorems, and this is very important. However, each statement in proof has to be established using only logic. Very often we hear or said statement like those two angles must be 90°, because the two lines look as if they are perpendicular to each other. Beware of being deceived by this type of reasoning.

Theorem-15.4 : The sum of three interior angles of a triangle is 180°.



Proof : Consider a triangle ABC.

We have to prove that

$\angle ABC + \angle BCA + \angle CAB = 180°$

Construct a line CE parallel to BA through C and produce line BC to D.

CE is parallel to BA and AC is transversal.

So, $\angle CAB = \angle ACE$, which are alternate angles. ..... (1)

Similarly, $\angle ABC = \angle DCE$ which are corresponding angles. ..... (2)

adding eq. (1) and (2) we get

$$\angle CAB + \angle ABC = \angle ACE + \angle DCE \quad ..... (3)$$

add $\angle BCA$ on both the sides.

We get, $\angle ABC + \angle BCA + \angle CAB = \angle DCE + \angle BCA + \angle ACE$ ..... (4)

But $\angle DCE + \angle BCA + \angle ACE = 180°$, since they form a straight angle...... (5)

Hence, $\angle ABC + \angle BCA + \angle CAB = 180°$

Now, we see how each step has been logically connected in the proof.

Step-1: Our theorem is concerned with a property of triangles. So we begin with a triangle ABC.

Step-2: The construction of a line CE parallel to BA and producing BC to D is a vital step to proceed so that to be able to prove the theorem.

Step-3: Here we conclude that $\angle CAB = \angle ACE$ and $\angle ABC = \angle DCE$, by using the fact that CE is parallel to BA (construction), and previously known theorems, which states that if two parallel lines are intersected by a transversal, then the alternate angles and corresponding angles are equal.

Step-4: Here we use Euclid's axiom which states that "if equals are added to equals, the wholes are equal" to deduce $\angle ABC + \angle BCA + \angle CAB = \angle DCE + \angle BCA + \angle ACE$.

That is, the sum of three interior angles of a triangle is equal to the sum of angles on a straight line.

Step-5: Here in concluding the statement we use Euclid's axiom which states that "things which are equal to the same thing are equal to each other" to conclude that

$$\angle ABC + \angle BCA + \angle CAB = \angle DCE + \angle BCA + \angle ACE = 180°$$

This is the claim made in the theorem we set to prove.

You now prove theorem-15.2 and 15.3 without analysing them.

Theorem-15.5 : The product of two odd natural numbers is odd.

Proof : Let x and y be any two odd natural numbers.

We want to prove that xy is odd.

Since x and y are odd, they can be expressed in the form $x = (2m - 1)$, $y = 2n - 1$ (for some natural number m, n)

Then, $xy = (2m - 1)(2n - 1)$

$= 4mn - 2m - 2n + 1$

$= 4mn - 2m - 2n + 2 - 1$

$= 2(2mn - m - n + 1) - 1$

Let $2mn - m - n + 1 = l$, any natural number, replace it in the above equation.

$= 2l - 1, l \in N$

This is definitely an odd number.

Theorem-15.6 : The product of any two consecutive even natural numbers is divisible by 4.

Any two consecutive even number will be of the form $2m$, $2m + 2$, for some natural number n. We have to prove that their product $2m(2m + 2)$ is divisible by 4. (Now try to prove this yourself).

We conclude this chapter with a few remarks on the difference between how mathematicians discover results and how formal rigorous proofs are written down. As mentioned above, each proof has a key intiative idea. Intution is central to a mathematicians' way of thinking and discovering results. A mathematician will often experiment with several routes of thought, logic and examples, before she/he can hit upon the correct solution or proof. It is only after the creative phase subsides that all the arguments are gathered together to form a proper proof.

We have discussed both inductive reasoning and deductive reasoning with some examples.

It is worth mentioning here that the great Indian mathematician Srinivasa Ramanujan used very high levels of intuition to arrive at many of his statements, whch he claimed were true. Many of these have turned out to be true and as well as known theorems.

Exercise - 15.4

1. State which of the following are mathematical statements and which are not? Give reason.
   i. She has blue eyes
   ii. $x + 7 = 18$
   iii. Today is not Sunday.
   iv. For each counting number x, $x + 0 = x$
   v. What time is it?
2. Find counter examples to disprove the following statements:
   i. Every rectangle is a square.
   ii. For any integers x and y, $\sqrt{x^2 + y^2} = x + y$
   iii. If n is a whole number then $2n^2 + 11$ is a prime.
   iv. Two triangles are congruent if all their corresponding angles are equal.
   v. A quadrilateral with all sides are equal is a square.
3. Prove that the sum of two odd numbers is even.
4. Prove that the product of two even numbers is an even number.

5. Prove that if x is odd, then x^2 is also odd.

6. Examine why they work ?

i. Choose a number. Double it. Add nine. Add your original number. Divide by three. Add four. Subtract your original number. Your result is seven.

ii. Write down any three-digit number (for example, 425). Make a six-digit number by repeating these digits in the same order (425425). Your new number is divisible by 7, 11, and 13.

What We Have Discussed?

1. The sentences that can be judged on some criteria, no matter by what process for their being true or false are statements.
2. Mathematical statements are of a distinct nature from general statements. They can not be proved or justified by getting evidence while they can be disproved by finding a counter example.
3. Making mathematical statements through observing patterns and thinking of the rules that may define such patterns.

   A hypothesis is a statement of idea which gives an explanation to a sense of observation.
4. A process which can establish the truth of a mathematical statement based purely on logical arguments is called a mathematical proof.
5. Axioms are statements which are assumed to be true without proof.
6. A conjecture is a statement we believe is true based on our mathematical intution, but which we are yet to prove.
7. A mathematical statement whose truth has been established or proved is called a theorem.
8. The prime logical method in proving a mathematical statement is deductive reasoning.
9. A proof is made up of a successive sequence of mathematical statements.
10. Begining with given (Hypothesis) of the theorem and arrive at the conclusion by means of a chain of logical steps is mostly followed to prove theorems.
11. The proof in which, we start with the assumption contrary to the conclusion and arriving at a contradiction to the hypothesis is another way that we establish the original conclusion is true is another type of deductive reasoning.
12. The logical tool used in establishment the truth of an unambiguious statements to deductive reasoning.
13. The resoning which is based on examining of variety of cases or sets of data discovering pattern and forming conclusion is called Inductive reasoning.

Answers

EXERCISE 1.1

1. a. $-5, \frac{22}{7}, \frac{-2013}{2014}$

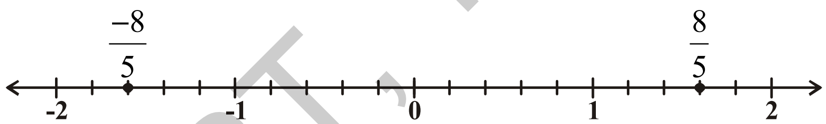
   b. A number which can be written in the form $\frac{p}{q}$ where $q \neq 0$; p, q are integers, called a rational number.

2. (i) $\frac{3}{7}$ (ii) 0 (iii) -5

   (iv) 7 (v) -3

3. $\frac{3}{2}, \frac{5}{4}, \frac{11}{8}, \frac{21}{16}, \frac{53}{32}$

4. $\frac{19}{30}, \frac{37}{60}, \frac{77}{120}$

5. 

6. I. (i) 0.242 (ii) 0.708 (iii) 0.4 (iv) 28.75

   II. (i) $0.\overline{6}$ (ii) $-0.69\overline{4}$ (iii) $3.\overline{142857}$ (iv) $1.\overline{2}$

7. (i) $\frac{9}{25}$ (ii) $\frac{77}{5}$ (iii) $\frac{41}{4}$ (iv) $\frac{13}{4}$

8. (i) $\frac{5}{9}$ (ii) $\frac{35}{9}$ (iii) $\frac{4}{11}$ (iv) $\frac{563}{180}$

9. (i) Yes (ii) No (iii) Yes (iv) No

EXERCISE - 1.2

1. (i) Irrational (ii) Rational (iii) Irrational

   (iv) Rational (v) Rational (vi) Irrational

2. Rational numbers : -1, $\frac{13}{7}$, 1.25, $21.\bar{8}$, 0

Irrational numbers : $\sqrt{2}$, $\sqrt{7}$, π, 2.131415....., 1.1010010001.....

3. $\frac{\sqrt{5}}{3}$, infinite solutions

4. 0.71727374....., 0.761661666..... 5. $\sqrt{5} = 2.236$

6. 2.645751 8. $\sqrt{5}, \sqrt{6}$

9. (i) True (ii) True (iii) True ($\sqrt{3}$) (iv) True ($\sqrt{9}$)

(v) True (vi) False $\left(\frac{3}{7}\right)$

Exercise - 1.4

1. (i) $10 + 5\sqrt{5} + 2\sqrt{7} + \sqrt{35}$ (ii) 20

(iii) $10 + 2\sqrt{21}$ (iv) 4

2. (i) Irrational (ii) Irrational (iii) Irrational (iv) Rational

(v) Irrational (vi) Irrational (vii) Rational

3. (i) Irrational (ii) Rational (iii) Irrational (iv) Irrational

(v) Irrational (vi) Rational

4. π is an irrational number, but not a surd.

5. (i) $\frac{3-\sqrt{2}}{7}$ (ii) $\sqrt{7}+\sqrt{6}$ (iii) $\frac{\sqrt{7}}{7}$ (iv) $3\sqrt{2}+2\sqrt{3}$

6. (i) $17-12\sqrt{2}$ (ii) $6-\sqrt{35}$ (iii) $\frac{3\sqrt{2}+2\sqrt{3}}{6}$ (iv) $\frac{9\sqrt{15}-3\sqrt{10}-3\sqrt{21}+\sqrt{14}}{25}$

7. 0.328 8. (i) 2 (ii) 2 (iii) 5 (iv) 64 (v) 9 (vi) $\frac{1}{6}$ 9. -8

10. (i) a = 5, b = 2 (ii) $a = \frac{-19}{7}$, $b = \frac{5}{7}$ 11. $\sqrt{6}+\sqrt{5}$

Exercise - 2.1

1. (i) 5 (ii) 2 (iii) 0 (iv) 6

(v) 2 (vi) 1

2. (i) Polynomial (ii) Polynomial (iii) No, because this polynomial has two variables.
(iv) Not polynomial because exponent is negative.
(v) Not polynomial because exponent of x is not a non negative integers.
(vi) Not polynomial in one variable because it has two variables.

3. (i) 1 (ii) -1 (iii) $\sqrt{2}$ (iv) 2
(v) $\frac{\pi}{2}$ (vi) $\frac{-2}{3}$ (vii) 0 (viii) 0

4. (i) Quadratic (ii) Cubic (iii) Quadratic (iv) Linear
(v) Linear (vi) Quadratic

5. (i) True (ii) False (iii) True (iv) False
(v) True (vi) True

Exercise - 2.2

1. (i) 3 (ii) 12 (iii) 9 (iv) $\frac{3}{2}$

2. (i) 1, 1, 3 (ii) 2, 4, 4 (iii) 0, 1, 8 (iv) -1, 0, 3
(v) 2, 0, 0

3. (i) Yes (ii) No (iii) Yes (iv) No, Yes
(v) Yes (vi) Yes (vii) Yes, No (viii) Yes, No

4. (i) -2 (ii) 2 (iii) $\frac{-3}{2}$ (iv) $\frac{3}{2}$
(v) 0 (vi) 0 (vii) $\frac{-q}{p}$

5. $a = \frac{-2}{7}$ 6. $a = 1, b = 0$

Exercise - 2.3

1. (i) 0 (ii) $\frac{27}{8}$ (iii) 1
(iv) $-\pi^3+3\pi^2-3\pi+1$ (v) $\frac{-27}{8}$

2. $5p$ 3. Not a factor, as remainder is 5 4. -3 5. $\frac{-13}{3}$

6. $\frac{-13}{3}$ 7. 8 8. $\frac{21}{8}$ 9. $a = -7, b = -12$

EXERCISE - 2.4

1. (i) Yes (ii) No (iii) No (iv) No
2. (i) Yes (ii) Yes (iii) Yes (iv) Yes
   (v) Yes
7. (i) $(x-1)(x+1)(x-2)$ (ii) $(x+1)(x+1)(x-5)$
   (iii) $(x+1)(x+2)(x+10)$ (iv) $(y+1)(y+1)(y-1)$
9. a = 3 10. $(y-2)(y+3)$

EXERCISE - 2.5

1. (i) $x^2 + 7x + 10$ (ii) $x^2 - 10x + 25$
   (iii) $9x^2 - 4$ (iv) $x^4 - \dfrac{1}{x^4}$ (v) $1 + 2x + x^2$
2. (i) 9999 (ii) 998001 (iii) $\dfrac{9999}{4} = 2499\dfrac{3}{4}$
   (iv) 251001 (v) 899.75
3. (i) $(4x+3y)^2$ (ii) $(2y-1)^2$ (iii) $\left(2x+\dfrac{y}{5}\right)\left(2x-\dfrac{y}{5}\right)$
   (iv) $2(3a+5)(3a-5)$ (v) $(x+3)(x+2)$
   (vi) $3(P-6)(P-2)$
4. (i) $x^2 + 4y^2 + 16z^2 + 4xy + 16yz + 8xz$
   (ii) $8a^3 - 36a^2b + 54ab^2 - 27b^3$
   (iii) $4a^2 + 25b^2 + 9c^2 - 20ab - 30bc + 12ac$
   (iv) $\dfrac{a^2}{16} + \dfrac{b^2}{4} + 1 - \dfrac{ab}{4} - b + \dfrac{a}{2}$
   (v) $p^3 + 3p^2 + 3p + 1$ (vi) $x^3 - 2x^2y + \dfrac{4}{3}xy^2 - \dfrac{8}{27}y^3$
5. (i) $(-5x + 4y + 2z)^2$ (ii) $(3a + 2b - 4c)^2$
6. 29
7. (i) 9,70,299 (ii) 10,61,208 (iii) 99,40,11,992 (iv) 100,30,03,001
8. (i) $(2a+b)^3$ (ii) $(2a-b)^3$ (iii) $(1-4a)^3$ (iv) $\left(2p-\dfrac{1}{5}\right)^3$
10. (i) $(3a+4b)(9a^2 - 12ab + 16b^2)$ (ii) $(7y-10)(49y^2 + 70y + 100)$
11. $(3x+y+z)(9x^2 + y^2 + z^2 - 3xy - yz - 3xz)$

14. (i) -630 (ii) 16380 (iii) $\frac{-5}{12}$ (iv) -0.018

15. (i) $(2a + 3)(2a - 1)$ (ii) $(5a - 3)(5a - 4)$

16. (i) $3x(x - 2)(x + 2)$ (ii) $4(3y + 5)(y - 1)$

Exercise - 3.1

1. (i) 3 (ii) 13 (iii) Both have 6 faces (iv) 180°
   (v) Point, Plane, Line
2. a) False b) True c) True d) True
   e) True 7. Infinite 8. Lines intersect on the side of the angle less than 180^0
9. $\angle 1 = \angle 2$

Exercise - 4.1

2. (i) Reflex angle (ii) Right angle (iii) Acute angle
3. (i) False (ii) True (iii) False (iv) False
   (v) True (vi) True (vii) False (viii) True
4. (i) 270° (ii) 180° (iii) 210°

Exercise - 4.2

1. $x = 36°$ $y = 54°$ $z = 90°$
2. (i) $x = 23°$ (ii) $x = 59°$ (iii) $x = 20°$ (iv) $x = 8°$
3. $\angle BOE = 30°$; Reflex angle of $\angle COE = 250°$
4. $\angle C = 126°$
8. $\angle XYQ = 122°$ Reflex $\angle QYP = 302°$

Exercise - 4.3

2. $x = 126°$
3. $\angle AGE = 126°$ $\angle GEF = 36°$ $\angle FGE = 54°$
4. $\angle QRS = 60°$ 5. $\angle ACB = z° = x° + y°$
6. $a = 40°$; $b = 100°$
7. (i) $\angle 3, \angle 5, \angle 7, \angle 9, \angle 11, \angle 13, \angle 15$
   (ii) $\angle 4, \angle 6, \angle 8, \angle 10, \angle 12, \angle 14, \angle 16$

8. $x = 60°$ $\quad y = 59°$

9. $x = 40°$ $\quad y = 40°$

10. $x = 60°$ $\quad y = 18°$

11. $x = 63°$ $\quad y = 11°$

13. $x = 50°$ $\quad y = 77°$

15. (i) $x = 36°$; $y = 108°$ (ii) $x = 35°$ (iii) $x = 29°$

16. $\angle 1 = \angle 3 = \angle 5 = \angle 7 = 80°$; $\angle 2 = \angle 4 = \angle 6 = \angle 8 = 100°$

17. $x = 20°$ $\quad y = 60°$ $\quad z = 120°$

18. $x = 55°$ $\quad y = 35°$ $\quad z = 125°$

19. (i) $x = 140°$ (ii) $x = 100°$ (iii) $x = 250°$

Exercise - 4.4

1. (i) $x = 110°$ (ii) $z = 130°$ (iii) $y = 80°$

2. $\angle 1 = 60°$ $\quad$ 3. $x = 35°, y = 51°$ $\quad$ 5. $x = 50°$ $\quad y = 20°$

6. $x = 70°$ $\quad y = 40°$ $\quad$ 7. $x = 30°$ $\quad y = 75°$

8. $\angle PRQ = 65°$ $\quad$ 9. $\angle OZY = 32°$; $\angle YOZ = 121°$

10. $\angle DCE = 92°$ $\quad$ 11. $\angle SQT = 60°$ $\quad$ 12. $z = 60°$

13. $x = 37°$ $\quad y = 53°$ $\quad$ 14. $\angle A = 50°$; $\angle B = 75°$

15. (i) 78° (ii) $\angle ADE = 67°$ (iii) $\angle CED = 78°$

16. (i) $\angle ABC = 72°$ (ii) $\angle ACB = 72°$
    (iii) $\angle DAB = 27°$ (iv) $\angle EAC = 32°$

17. $x = 96°$ $\quad y = 120°$

Exercise - 5.1

1. (i) Water Tank (ii) Mr. 'J' house
   (iii) In street-2, third house on right side while going in east direction.
   (iv) In street 4, first building on right side while going in east direction.
   (v) In street 4, the third building on left side while going in east direction

Exercise - 5.2

1. (i) Q_2 (ii) Q_4 (iii) Q_1 (iv) Q_3
   (v) Y-axis (vi) X-axis (vii) X-axis (viii) Y-axis

2. (i) abscissa : 4, ordinate : -8 (ii) abscissa : -5, ordinate : 3 (iii) abscissa : 0, ordinate : 0 (iv) abscissa : 5, ordinate : 0

(v) abscissa : 0
ordinate : -8

3. (ii) (0, 13) : Y-axis (iv) (-2, 0) : X-axis
(v) (0, -8) : Y-axis (vi) (7, 0) : X-axis
(vii) (0, 0) : on both the axis.

4. (i) -7 (ii) 7 (iii) R (iv) P
(v) 4 (vi) -3

5. (i) False (ii) True (iii) True (iv) False (v) False (vi) False

EXERCISE - 5.3

2. No. (5, -8) lies in Q_4 and (-8, 5) lies in Q_2
3. All given points lie on a line parallel to Y-axis at a distance of 1 unit.
4. All points lie on a line parallel to X-axis at a distance of 4 units.
5. 12 Sq.units. 6. 8 Sq. units

EXERCISE - 6.1

1. (i) $a = 8$ $b = 5$ $c = -3$
(ii) $a = 28$ $b = -35$ $c = 7$
(iii) $a = 93$ $b = 15$ $c = -12$
(iv) $a = 2$ $b = 5$ $c = 0$
(v) $a = \frac{1}{3}$ $b = \frac{1}{4}$ $c = -7$
(vi) $a = \frac{3}{2}$ $b = 1$ $c = 0$
(vii) $a = 3$ $b = 5$ $c = -12$

2. (i) $a = 2$ $b = 0$ $c = -5$
(ii) $a = 0$ $b = 1$ $c = -2$

(iii) $a = 0$ $b = \frac{1}{7}$ $c = -3$

(iv) $a = 1$ $b = 0$ $c = \frac{14}{13}$

3. (i) $x + y = 34$ (ii) $x - 2y - 10 = 0$ or $2x - y + 10 = 0$

(iii) $x - 2y - 10 = 0$ (iv) $2x + 15y - 100 = 0$

(v) $x + y - 200 = 0$ (vi) $x + y - 11 = 0$

Exercise - 6.2

2. (i) $(0, -34); (\frac{17}{4}, 0)$ (ii) $(0, 3) ; (-7, 0)$

(iii) $(0, \frac{3}{2}) ; (\frac{-3}{5}, 0)$

3. (i) Not a solution (ii) Solution (iii) Solution

(iv) Not a solution (v) Not a solution

4. k = 7   5. $\alpha = \frac{8}{5}$ 6. 3

Exercise - 6.3

2. (i) Yes (ii) Yes

3. 3

4. (i) 6 (ii) -5

5. (i) $(\frac{3}{2}, 3)$ (ii) $(-3, 6)$

6. (i) (2, 0) : (0, -4) (ii) (-8, 0) ; (0, 2)

(iii) (-2, 0) ; (0, -3)

7. $x + y = 1000$ 8. $x + y = 5000$ 9. $f = 6a$ 10. 39.2 units

Exercise - 6.4

1. $5x = 3y$; 2000 ; 480 (No. of voters who cast their vote = x, Total no. of voters = y)

2. x - y = 25; 50; 15 (Father age = x, Rupa's age = y)

3. $y = 8x + 15$, 6km, ₹ 63   4. $x + 4y = 27$; 11, 5

5. $y = 10x + 30$; 60; 90; 5 hr. (No. of hours = x ; Parking charges = y)

6. d = 60 t (d = distance, t = time); 90 km.; 120 km.; 210 km.

7. $y = 8x$; $\frac{3}{2}$ or $1\frac{1}{2}$; 12

8. $y = \frac{5}{7}x$ (Quantity of mixture = x; Quantity of milk = y) ; 20

9. (ii) 86° F (iii) 35° C (iv) -40

Exercise - 6.5

4. (i) $y = -3$ (ii) $y = 4$ (iii) $y = -5$ (iv) $y = 4$

5. (i) $x = -4$ (ii) $x = 2$ (iii) $x = 3$ (iv) $x = -4$

Exercise - 7.4

6. 7

7. No.

Exercise - 8.1

1. (i) True (ii) True (iii) False (iv) True
   (v) False (vi) False

2. (a) Yes, No, No, No, No (b) No, Yes, Yes, Yes, Yes
   (c) No, Yes, Yes, Yes, Yes (d) No, Yes, Yes, Yes, Yes
   (e) No, Yes, Yes, Yes, Yes (f) No, Yes, Yes, Yes, Yes
   (g) No, No, No, Yes, Yes (h) No, No, Yes, No, Yes
   (i) No, No, No, Yes, Yes (j) No, No, Yes, No, Yes.

4. Four angles = 36°, 72°, 108°, 144°

Exercise - 8.3

1. Angles of parallelogram = 73°, 107°, 73°, 107°

2. Angles of parallelogram = 68°, 112°, 68°, 112°

Exercise - 8.4

1. BC = 8 cm.

EXERCISE - 9.1

1.

Marks	5	6	7	8	9	10
Frequency (f)	5	6	8	12	9	5

2.

Blood Group	A	B	AB	O
Frequency (f)	10	9	2	15

Most common blood group = O ; Most rarest blood group = AB

3.

No. of Heads	0	1	2	3
Frequency (f)	3	10	10	7

4.

Options	A	B	C
Frequency (f)	19	36	10

Total appropriate answers = 65
Majority of people's opinion = B (Prohibition in public place only)

5.

Type of Vehicles	Car	Bikes	Autos	Cycles
No. of Vehicles (f)	25	45	30	40

6. Scale : on X-axis = 1 cm. = 1 class interval
on X-axis = 1 cm. = 10 number of students

Class	I	II	III	IV	V	VI
No. of students (f)	40	55	65	50	30	15

7.

Marks (Class interval)	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
No. of students (f)	1	4	3	7	7	7	1	0

8.

Electricity Bills (in ₹) (Class Interval)	No. of Houses (f)
150 - 225	4
225 - 300	3
300 - 375	7
375 - 450	7
450 - 525	0
525 - 600	1
600 - 675	1
675 - 750	2

9.

Life time (in years) (Class Interval)	2-2.5	2.5-3.0	3.0-3.5	3.5-4.0	4.0-4.5	4.5-5.0
No. of Batteries	2	6	14	11	4	3

EXERCISE - 9.2

1. $\bar{x} = 85$ 2. $\bar{x} = 1.71$ 3. K = 10
4. $\bar{x} = 17.7$
5. (i) ₹ 359, ₹ 413, ₹ 195, ₹ 228, ₹ 200, ₹ 837
   (ii) ₹444 saving per school.
6. Boy's height = 147 cm. ; Girl's height = 152 cm.
7. $\bar{x}$ = 11.18 ; Mode = 5 ; Median = 10
8. $\bar{x}$ = 80 ; Median = 75 ; Mode = 50
9. 37 kgs 10. ₹11.25, Median = ₹ 10; Mode = ₹ 10
11. $1^{st} = 2$; $2^{nd} = 6$; $3^{rd} = 19$; $4^{th} = 33$

EXERCISE - 10.1

1. (i) 64 cm^2, 96 cm^2 (ii) 140 cm^2, 236 cm^2
2. 3375 m^3 3. 330 m^3 4. 8 cm.
5. (i) 4 times of original area (ii) 9 times of original area (iii) n^2 times
6. 60 cm^3 7. 48 m^3 8. 3750000 liters

EXERCISE - 10.2

1. 6.90 m^2 2. 176 cm^2; 253 cm^2
3. r = 7.5 cm. 4. h = 2.5 m.
5. (i) 968 cm^2 (ii) 1064.8 cm^2 (iii) 2038.08 cm^2
6. ₹ 5420. 80 7. 1584 m^2
8. (i) 110 m^2 (ii) ₹4400
9. (i) 87.12 m^2 (ii) 95.04 m^2 10. 517.44 liters 11. h = 20 cm.

EXERCISE - 10.3

1. h = 6 cm. 2. h = 9 cm.
3. (i) 7 cm. (ii) 462 cm^2 4. 1232 cm^3

5. $1018.3\ cm^3$ 6. ₹7920, 15m 7. $3394\frac{2}{7}\ cm^3$

8. $241.84\ m^2$ (approximate) 9. 63m 10. $6135.8\ cm^2$ 11. 24.7 min

12. 60π sq. units.

Exercise - 10.4

1.	$154\ cm^2$; $179.67\ cm^3$			2.	$3054.86\ cm^3$		
3.	$616\ cm^2$	4.	$6930\ cm^2$	5.	4 : 9 ; 8 : 27		
6.	$942 cm^2$	7.	1 : 4	8.	441 : 400	9.	55 gms or 0.055 kg
10.	5 cm.	11.	0.303 liters	12.	No. of bottles = 9		

Exercise - 11.1

1. $19.5\ cm^2$ 2. $114\ cm^2$ 3. $36\ cm^2$

Exercise - 11.2

1. 8.57 cm 2. 6.67 cm

Exercise - 12.1

1. (i) Radius (ii) Diameter (iii) Minor arc
(iv) Chord (v) Major arc (vi) Semi-circle
(vii) Chord (viii) Minor segment

2. (i) True (ii) True (iii) True (iv) False
(v) False (vi) True (vii) True

Exercise - 12.2

1. 90° 2. 48°, 84° 3. Yes

Exercise - 12.4

1.	130°	2.	40°	3.	60°, 120°	4.	5cm.
5.	6 cm.	6.	4 cm.	8.	70°, 55°, 55°		

EXERCISE 12.5

1. (i) $x° = 75°$; $y° = 75°$ (ii) $x° = 70°$; $y° = 95°$
   (iii) $x° = 90°$; $y° = 40°$
5. (a), (b), (c), (e), (f) = Possible ; (d) = Not possible

EXERCISE - 14.1

1. (a) 1, 2, 3, 4, 5 and 6 (b) Yes (c) $\frac{1}{3}$
2. (a) $\frac{45}{100}$; $\frac{55}{100}$ (b) 1
3. (a) Red (b) Yellow (c) Blue and Green (d) No chance
   (e) No (It is random experiment)
4. (a) No.
   (b) P (green) = $\frac{5}{12}$; P (blue) = $\frac{1}{4}$; P (red) = $\frac{1}{6}$; P (yellow) = $\frac{1}{6}$
5. (a) P(E) = $\frac{5}{26}$ (b) P(E) = $\frac{5}{13}$ (c) 1 (d) $\frac{21}{26}$
6. P(E) = $\frac{7}{11}$
7. (i) P = $\frac{61}{806}$ (ii) P = $\frac{45}{146}$ (iii) P = $\frac{261}{400}$ 8. $\frac{3.43}{16}$

EXERCISE - 15.1

1. (i) Always false. There are minimum 28 days in a month. Usually we have months of 30 and 31 days.
   (ii) Ambiguous. In a given year, Makara Sankranthi may or may not fall on friday.
   (iii) Ambiguous. At some time in winter, there can be a possibility that Hyderabad have 2°C temperature.
   (iv) True, to the known fact, so far we can say this but it can be changed if scientists find evidances of life on other planets.
   (v) Always false. Dogs cannot fly.
   (vi) Ambiguous. In a leap year, February has 29 days.
2. (i) False, the sum of the interior angles of a quadrilateral is 360°.
   (ii) True - eg. all negative numbers.

(iii) True- Rhombus has opposite side parallel to each other therefore rhombus is parallelogram.

(iv) True

(v) No, all square number can not be written as a sum of two odd numbers, eg. 9 = 4+5 (But we can write all square numbers as a sum of odd, eg. 9 = 1 + 3 + 5 numbers)

3. (i) Only natural number

(ii) Two time a natural number is always even.

[eg. $2 \times \frac{5}{2} = 5$ (odd number)]

(iii) For any $x > 1$, $3x + 1 > 4$ (iv) For any $x \geq 0$, $x^3 \geq 0$

(v) In an equilateral triangle, a median is also an angle bisector.

4. Take any negative number $x = -2, y = -3$

$-2 > -3$ (Given)

$x^2 = -2 \times -2 = 4$ (here $x^2 < y^2$)

$y^2 = -3 \times -3 = 9$

Exercise - 15.2

1. (i) Jeevan is mortal

(ii) No, X could be any other state person lke marathi, gujarati, punjabi etc.

(iii) Gulag has red tongue.

(iv) All smarts need not be a president. Here we have given only that all presidents are smart. There could be some other people like some of the teachers, students who are smart too.

2. You need to turn over B and 8. If B has an even number on the other side, then the rule has been broken. Similarly, if 8 has a consonant on the other side, then the rule has been broken.

3. The answer is 35.

- Statement 'a' does not help because by following the other clues you can tell that you need more than on digit.
- Statement 'b' does not help because the one digit has to be larger than the tens-digit and the only multiple of 7 and 10 is 70 and 0 is smaller than the 7.
- Statement 'c' helps because being a multiple of 7 concels out a lot of numbers that could have been possibilities.
- Statement 'd' helps because being an odd number it too ancels out a lot of other possibilities.
- Statement 'e' does not help because the only multiple of 7 and 11 is 77 and the ones digit has to bigger than the tens digit.
- Statement 'g' helps because by using it there will be few numbers left.
- Statement 'h' helps by using it only 35 remains.

So - 3, 4, 7 and 8 helps and they only are enough to get the number.

Exercise - 15.3

1. (i) The possible three conjuctures are:

   a) The product of any three consecutive odd number is odd.

   b) The product of any three consecutive odd number is divisible by 3.

   c) The sum of all the digits present in product of three consecutive odd numbers is even.

   (ii) The possible three conjuctures are:

   a) The sum of any three consecutive number is always even.

   b) The sum of any three consecutive number is always divided by 3.

   c) The sum of any three consecutive number is always divided by 6.

4. $111111^2 = 12345654321$ $\quad$ $1111111^2 = 1234567654321$

   Conjecture is true

6. Conjecture is false because you can not find a composite number for $x = 41$.

Exercise - 15.4

1. (i) No $\quad$ (ii) Yes $\quad$ (iii) No

   (iv) Yes $\quad$ (v) No

2. (i) A rectangle has equal angles but may not be a square.

   (ii) For $x = 2; y = 3$, the statement is not true.
   (It is only true for $x = 0; y = 1$ or $x = 0, y = 0$)

   (iii) For $n = 11$, $2n^2 + 11 = 253$ which is not a prime number.

   (iv) You can give any two triangles with the same angles but of different sides.

   (v) A rhombus has equal sides but may not be a square.

3. Let x and y be two odd numbers. Then $x = 2m + 1$ for some natural number m and $y = 2n + 1$ for some natural number n.

   $x + y = 2(m + n + 1)$. Therefore, $x + y$ is divisible by 2 and is even.

4. Let $x = 2m$ and $y = 2n$

   Product $xy = (2m)(2n)$

   $= 4mn$

6. (i) Let your original number be n. Then we are doing the following operations:

$$n \to 2n \to 2n+9 \to +n = 3n+9 \to \frac{3n+9}{3} = n+3 \to n+3+4 = n+7 \to n+7-n = 7$$

   (ii) Note that $7 \times 11 \times 13 = 1001$. Take any three digit number say, *abc*. Then abc $\times$ 1001 = *abcabc*. Therefore, the six digit number *abcabc* is divisible by 7, 11 and 13.

SYLLABUS

Number System (10 hrs) **(i) Real numbers**	**(i) Real numbers** • Review of representation of natural numbers, integers, and rational numbers on the number line. • Representation of terminating / non terminating re curring decimals, on the number line through successive magnification. • Rational numbers as recurring / terminating decimals. • Finding the square root of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ correct to 6-decimal places by division method • Examples of nonrecurring / non terminating decimals such as 1.01011011101111— 1.12112111211112— and $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ etc. • Existence of non-rational numbers (irrational numbers) such as $\sqrt{2}$, $\sqrt{3}$ and their representation on the number line. • Existence of each real number on a number line by using Pythogorian result. • Concept of a Surd. • Rationalisation of surds • Square root of a surd of the form $a+\sqrt{b}$
Algebra (20 hrs) **(i) Polynomials** **(ii) Linear Equations in Two Variables**.	**(i) Polynomials** • Definition of a polynomial in one variable, its coefficients, with examples and counter examples, its terms, zero polynomial. • Constant, linear, quadratic, cubic polynomials; monomials, binomials, trinomials. Zero / roots of a polynomial / equation. • State and motivate the Remainder Theorem with examples and analogy to positive integers (motivate). • Statement and verification of the Factor Theorem. Factorization of $ax^2 + bx + c$, $a \neq 0$ where a, b, c are real numbers and factorization of cubic polynomials using the Factor Theorem.

	• Recall of algebraic expressions and identities. • Further identities of the type: $(x+y+z)^2 \equiv x^2+y^2+x^2+2xy+2yz+2zx$ $(x \pm y)^3 \equiv x^3 \pm y^3 \pm 3xy\,(x \pm y)$ $x^3+y^3+z^3-3xyz \equiv (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$ $x^3+y^3 \equiv (x+y)(x^2-xy+y^2)$ $x^3-y^3 \equiv (x-y)(x^2+xy+y^2)$ and their use in factorization of polynomials. **(ii) Linear Equations in Two Variables** • Recall of linear equations in one variable. • Introduction to the equation in two variables. • Solution of a linear equation in two variables • Graph of a linear equation in two variables. • Equations of lines parallel to x-axis and y-axis. • Equations of x-axis and y-axis.
Coordinate geometry **(5 hrs)**	**Coordinate geometry** • Cartesian system • Plotting a point in a plane if its co-ordinates are given.
Geometry (40 hrs) **(i) The Elements of Geometry** **(ii) Lines and Angles** **(iii) Triangles** **(iv) Quadrilaterals** **(v) Area** **(vi) Circles** **(vii) Geometrical Constructions**	**(i) The Elements of Geometry** • History – Euclid and geometry in India. Euclid's method of formalizing observed phenomenon onto rigorous mathematics with definitions, common / obvious notions, axioms / postulates, and theorems. The five postulates of Euclid. Equivalent varies of the fifth postulate. Showing the relationship between axiom and theorem. • Given two distinct points, there exists one and only one line through them. • (Prove) Two distinct lines cannot have more than one point in common.

(ii) Lines and Angles

- (Motivate) If a ray stands on a line, then the sum of the two adjacent angles so formed is 180^0 and the converse.
- (Prove) If two lines intersect, the vertically opposite angles are equal.
- (Motivate) Results on corresponding angles, alternate angles, interior angles when a transversal intersects two parallel lines.
- (Motivate) Lines, which are parallel to given line, are parallel.
- (Prove) The sum of the angles of a triangle is 180^0.
- (Motivate) If a side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.

(iii) Triangles

- (Motivate) Two triangles are congruent if any two sides and the included angle of one triangle is equal to any two sides and the included angle of the other triangle (SAS Congruence).
- (Prove) Two triangles are congruent if any two angles and the included side of one triangle is equal to any two angles and the included side of the other triangle (ASA Congruence).
- (Motivate) Two triangles are congruent if the three sides of one triangle are equal to three sides of the other triangle (SSS Congruence).
- (Motivate) Two right triangles are congruent if the hypotenuse and a side of one triangle are equal respectively to the hypotenuse and a side of the other triangle.
- (Prove) The angles opposite to equal sides of a triangle are equal.
- (Motivate) The sides opposite to equal angles of a triangle are equal.
- (Motivate) Triangle inequalities and relation between 'angle and facing side'; inequalities in a triangle.

(iv) Quadrilaterals

- (Prove) The diagonal divides a parallelogram into two congruent triangles.
- (Motivate) In a parallelogram opposite sides are equal and conversely.
- (Motivate) In a parallelogram opposite angles are equal and conversely.
- (Motivate) A quadrilateral is a parallelogram if one pair of its opposite sides are parallel and equal.
- (Motivate) In a parallelogram, the diagonals bisect each other and conversely.
- (Motivate) In a triangle, the line segment joining the mid points of any two sides is parallel to the third side and (motivate) its converse.

(v) Area

- Review concept of area, area of planar regions.
- Recall area of a rectangle.
- Figures on the same base and between the same parallels.
- (Prove) Parallelograms on the same base and between the same parallels have the same area.
- (Motivate) Triangles on the same base and between the same parallels are equal in area and its converse.

(vi) Circles

- Through examples, arrive at definitions of circle related concepts radius, circumference, diameter, chord, angle subtended by arc.
- (Prove) Equal chords of a circle subtend equal angles at the centre and (motivate) its converse.
- (Motivate) The perpendicular from the centre of a circle to a chord bisects the chord and conversely, the line drawn through the centre of circle to bisect a chord is perpendicular to the chord.

	• (Motivate) There is one and only one circle passing through three given non-collinear points. • (Motivate) Equal chords of a circle (or of congruent circles) are equidistant from the centre (s) and conversely. • (Prove) The angle subtended by am arc at the centre is double the angle subtended by it at any point on the remaining part of the circle. • (Motivate) Angles in the same segment of a circle are equal. • (Motivate) A line segment joining any two points subtends equal angles at two other points lying on the same side of it then the four points are concyclic. • (Motivate) The sum of the either pair of the opposite angles of a cyclic quadrilateral is 180^0 and its converse.
	(vii) Geometrical Constructions • Construction of a triangle given its base, sum / difference of the other two sides and one base angles. • Construction of a triangle when its perimeter and base angles are given. • Construct a circle segment containing given chord and given an angle.
Mensuration (15 hrs) **(i) Surface Areas and Volumes**	**(i) Surface Areas and Volumes** • Revision of surface area and volume of cube, cuboid. • Surface areas of cylinder, cone, sphere, hemi sphere. • Volume of cylinder, cone, sphere. (including hemi spheres) and right circular cylinders/ cones.
Statistics and Probability (15 hrs) **(i) Statistics** **(ii) Probability**	**(i) Statistics** • Revision of ungrouped and grouped frequency distributions. • Mean, Median and Mode of ungrouped frequency distribution (weighted scores).
	(ii) Probability • Feel of probability using data through experiments. Notion of chance in events like tossing coins, dice etc. • Tabulating and counting occurrences of 1 through 6 in a number of throws.

	• Comparing the observation with that for a coin. Observing strings of throws, notion of randomness. • Consolidating and generalizing the notion of chance in eventslike tossing coins, dice etc. • Visual representation of frequency outcomes of repeated throws of the same kind of coins or dice. • Throwing a large number of identical dice/coins together and aggregating the result of the throws to get large number of individual events. • Observing the aggregating numbers over a large number of repeated events.Comparing with the data fora coin. Observing strings of throws, notion of randomness.
Proofs in Mathematics (5 hrs) **(i) Proofs in Mathematics**	**(i) Proofs in Mathematics** • Mathematical statements, verifying them. • Reasoning Mathematics, deductive reasoning • Theorems, conjectures and axioms. • What is a mathematical proof.

Academic Standards

Academic standards are clear statements about what students must know and be able to do. The following are categories on the basis of which we lay down academic standards

Problem Solving

Using concepts and procedures to solve mathematical problems

(a) Kinds of problems:

Problems can take various forms- puzzles, word problems, pictorial problems, procedural problems, reading data, tables, graphs etc.

(b) Problem Solving - steps

- Reads problems
- Identifies all pieces of information/data
- Separates relevant pieces of information
- Understanding what concept is involved
- Recalling of (synthesis of) concerned procedures, formulae etc.
- Selection of procedure
- Solving the problem
- Verification of answers using theorems, problems based on theorems.

(c) Complexity:

The complexity of a problem is dependent on the following

- Making connections (as defined in the connections section)
- Number of steps in the problem
- Number of operations in the problem
- Context unraveling
- Nature of procedures

Reasoning Proof

- Reasoning between various steps
- Understanding and making mathematical generalizations and conjectures
- Understands and justifies procedures· Examining logical arguments.

- Understanding the notion of proof
- Uses inductive and deductive logic
- Testing mathematical conjectures

Communication

- Writing and reading, expressing mathematical notations (verbal and symbolic forms)

  Ex: $3 + 4 = 7$, $3 < 5$, $n_1+n_2= n_2+n_1$, Sum of angles $= 180^0$
- Creating mathematical expressions
- Explaining mathematical ideas in her own words like- a square is closed figure having four equal sides and all equal angles
- Explaining mathematical procedures like adding two digit numbers involves first adding the digits in the units place and then adding the digits at the tens place/ keeping in mind carry over.
- Explaining mathematical logic

Connections

- Connecting concepts within a mathematical domain- for example relating adding to multiplication, parts of a whole to a ratio, to division. Patterns and symmetry, measurements and space
- Making connections with daily life
- Connecting mathematics to different subjects
- Connecting concepts of different mathematical domains like data handling and arithmetic or arithmetic and space
- Connecting concepts to multiple procedures

Visualization & Representation

- Interprets and reads data in a table, number line, pictograph, bar graph, 2-D figures, 3-D figures, pictures
- Making tables, number line, pictograph, bar graph, pictures.
- Mathematical symbols and figures.

LEARNING OUTCOMES

The learner

- applies logical reasoning in classifying real numbers, proving their properties and using them in different situations.
- identifies/ classifies polynomials among algebraic expressions and factorises them by applying appropriate algebraic identities.
- relates the algebraic and graphical representations of a linear equation in one or two variables and applies the concept to daily life situations.
- identifies similarities and differences among different geometrical shapes.
- derives proofs of mathematical statements particularly related to geometrical concepts, like parallel lines, triangles, quadrilaterals, circles etc., by applying axiomatic approach and solves problems using them.
- finds areas of all types of triangles by using appropriate formulae and apply them in real life situations.
- constructs different geometrical shapes like bisectors of line segments, angles and triangles under given conditions and provides reasons for the processes of such constructions.
- develops strategies to locate points in a Cartesian plane.
- identifies and classifies the daily life situations in which mean, median and mode can be used.
- analyses data by representing it in different forms like, tabular form (grouped or ungrouped), bar graph, histogram (with equal and varying width and length), and frequency polygon, frequency curve and ogive curves
- calculates empirical probability through experiments and describes its use in words.
- derives formulae for surface areas and volumes of different solid objects like, cubes, cuboids, right circular cylinders/ cones, spheres and hemispheres and applies them to objects found in the surroundings.
- solves problems that are not in the familiar context of the child using above learning. These problems should include the situations to which the child is not exposed earlier.

Textbook - Overview

The Government of Telangana has decided to revise the curriculum of all the subjects based on State Curriculum Frame work (SCF - 2011) which recommends that children's life at schools must be linked to their life outside the school. Right to Education (RTE - 2009) perceives that every child who enters the school should acquire the necessary skills prescribed at each level upto the age of 14 years. The introduction of syllabus based on National Curriculum Frame Work - 2005 is every much necessary especially in Mathematics and Sciences at secondary level with a national perspective to prepare our students with a strong base of Mathematics and Science.

The strength of a nation lies in its commitment and capacity to prepare its people to meet the needs, aspirations and requirements of a progressive technological society.

The syllabus in Mathematics for three stages i.e. primary, upper primary and secondary is based on structural and spiral approaches. The teachers of secondary school Mathematics have to study the syllabus of classes 8 to 10 with this background to widen and deepen the understanding and application of concepts learnt by pupils in primary and upper primary stages.

The syllabus is based on the structural approach, laying emphasis on the discovery and understanding of basic mathematical concepts and generalisations. The approach is to encourage the pupils to participate, discuss and take an active part in the classroom processes.

The present text book has been written on the basis of curriculum and Academic standards emerged after a thorough review of the curriculum prepared by the SCERT.

- The syllabus has been divided broadly into six areas namely, Number System, Algebra, Geometry, Measuration, Statistics and Coordinate Geometry. Teaching of the topics related to these areas will develop the skills prescribed in academic standards such as problem solving, logical thinking, mathematical communication, representing data in various forms, using mathematics as one of the disciplines of study and also in daily life situations.

  The text book attempts to enhance this endeavor by giving higher priority and space to opportunities for contemplations. There is a scope for discussion in small groups and activities required for hands on experience in the form of 'Do this' and 'Try this'. Teacher's support is needed in setting the situations in the classroom.

Some special features of this text book are as follows

- The chapters are arranged in a different way so that the children can pay interest to all curricular areas in each term in the course of study.
- Teaching of geometry in upper primary classes was purely an intuition and to discover properties through measurements and paper foldings. Now, we have stepped into an axiomatic approach. Several attempts are made through illustrations to understand, defined, undefined terms and axioms and to find new relations called theorems as a logical consequence of the accepted axioms. Care has been taken to see that every theorem is provided initially with an activity for easy understanding of the proof of those theorems.
- Continuous Comprehension Evaluation Process has been covered under the tags of 'Try this' and 'Think, Discuss and Write'. Exercises are given at the end of each sub item of the chapter so that the teacher can assess the performance of the pupils throughout the chapter.
- Entire syllabus is divided into 15 chapters, so that a child can go through the content well in bit wise to consolidate the logic and enjoy the learning of mathematics.
- Some interesting and historical highlights are given under titles of Brain teasers, Do you know will certainly help the children for creative thinking.
- Colourful pictures, diagrams, readable font size will certainly help the children to adopt the contents and care this book as theirs.

Chapter (1) Real Numbers under the area number system and irrational numbers in detail.The child can visualise the rational and irrational numbers by the representation of them on number line. Some history of numbers is also added e.g value of to create interest among students. The representation of real numbers on the number line through successive magnification help to visualise the position of a real number with a non-terminating recuring decimal expansion.

Chapter (2) Polynomials and Factorisation under the area algebra dealt with polynomials in one variable and discussed about how a polynomial is diffierent from an algebraic expression. Facrtorisation of polynomials using remainder theorem and factors theorem is widely discussed with more number of illustrations . Factorisation of polynomials were discussed by splitting the middle term with a reason behind it. We have also discussed the factorisation of some special polynomials using the identities will help the children to counter various tuypes of factorisation.

Chapter (3) Linear equations in two variables under the same area will enable the pupil to discover through illustative examples,the unifying face of mathematical structure which is the ultimate objective of teaching mathematics as a system. This chapter links the ability of finding unknown with every day experience.

There are 7 chapters of Geometry i.e (**3, 4, 7, 8, 11, 12 and 13**) were kept in this book. All these chapters emphasis learning geometry using reasoning , intutive understanding and insightful personal experience of meanings. It helps in communicating and solving problems and obtaining new relations among various plane figures. Development geometry historically through centuries is given and discussed about Euclid's contribution in development of plane geometry through his collection "The Elements" . The activities and theorem were given on angles, triangles, quadrilaterals , circles and areas. It will develop induction, deduction, analytical thinking and logical reasoning. Geometrical constructions were presented insuch a way that the usage of an ungraduated ruler and a compass are necessary for a perfect construction of geometrical figures.

Chapter (5) deals with coordiante geometry as an alternate approach to Euclidean geometry by means of a coordinate system and associated algebra. Emphasis was given to plot ordered pairs on a cartesian plane (Graph) with a wide variety of illustratgive examples.

Chapter (9) statistics deals with importance of statistics , collection of statistical data i.e grouped and ungrouped , illustrative examples for finding mean, median and mode of a given data was discussed by taking daily life sitution.

Chapter (14) Probability is entirely a new chaper for secondary school students was introduced with wide variety of examples which deals with for finding probable chances of success in different fields. and mixed proportion problems with a variety of daily life situations.

Chapter (10) surface areas and volumes we discussed about finding curved (lateral) surface area, total surface area and volume of cylinder, cone and sphere. It is also discussed the relation among these solids in finding volumes and derive their formulae.

Chapter (15) Proofs in mathematics will help ;the students to understand what is a mathematical statement and how to prove a mathematical statement in various situations. We have also discussed about axiom , postulate, conjecture and the various stages in proving a theorem with illustrative examples. Among these 15 chapters the teacher has to Real Numbers, Polynomials and Factorisation, Co-ordinate geometry, Linear equation in two variables, Triangles, Quadrilaters and Areas under paper - I and the elements of Geometry, lines and angles, Statistics, Surface as a part of are volume, Circles Geometrical constructions and probability under paper - II.

The success of any course depends not so much on the syllabus as on the teacher and the teaching methods she employs. It is expected that all concerned with the improving of mathematics education would extend their full co operation in this endeavour.

Mere the production of good text books does not ensure the quality of education, unless the teachers transact the curriculum the way it is discussed in the text book. The involvement and participation of learner in doing the activities and problems with an understanding is ensured. Therefore it is expected that the teachers will bring a paradigm shift in the classroom process from mere solving the problems in the exercises routinely to the conceptual understanding, solving of problems with ingenity.

Text Book development committee

Highlight from History

"The Wonder of Discovery is especially keen in childhood"

How a child become Ramanujan a great mathematician of all the time?

Ramanujan

Srinivasa Ramanujan was the one who never lost his joy at learning something new. As a boy he impressed his classmates, senior students and teachers with his insight and intuition. One day in an Arithmetic class on division the teacher said that if three bananas were given to three boys, each boy would get a banana and he generalised this idea. Then Ramanujan asked "Sir, if no banana is distributed to no student will every one still get a banana?" Ramanujan's math ability won several friends to him. Once his senior student posed a problem "If $\sqrt{x}+y=7$ and $x+\sqrt{y}=11$, what are x and y". Immediately Ramanujan replied $x=9$ and $y=4$. His senior was impressed and became a good friend to him. In his school days, along with school homeworks, Ramanujan worked with some patterns out of his interest.

$$3=\sqrt{9}=\sqrt{1+8}$$
$$=\sqrt{1+(2\times 4)}$$
$$=\sqrt{1+2\sqrt{16}}$$
$$=\sqrt{1+2\sqrt{1+15}}$$
$$=\sqrt{1+2\sqrt{1+(3\times 5)}}$$

and so on ...

$$\frac{1}{4}+2=\left(1\frac{1}{2}\right)^2$$
$$\frac{1}{4}+(2\times 3)=\left(2\frac{1}{2}\right)^2$$
$$\frac{1}{4}+(2\times 3\times 5)=\left(5\frac{1}{2}\right)^2$$
$$\frac{1}{4}+(2\times 3\times 5\times 7)=\left(14\frac{1}{2}\right)^2$$

... and so on

Srinivasa Aaiyangar Ramanujan is undoubtedly the most celebrated Indian Mathematical genius. He was born in a poor family at Erode in Tamilnadu on December 22, 1887. Largely self taught, he feasted on **"Loney's Trigonometry"** at the age of 13, and at the age of 15, his senior friends gave him synopsis of Elementary results in pure and Applied mathematics by George Carr. He used to write his ideas and results on loose sheets. His filled note books are now famous as **"Ramanujan's Frayed note books"**. Though he had no qualifying degree, the university of Madras granted him a monthly Scholarship of Rs. 75 in 1913. He had sent papers of 120 theorems and formulae to great mathematican G.H. Hardy (Combridge University, London). They have recognised these as a worth piece and invited him to England. He had worked with Hardy and others and presented numerical theories on numbers, which include circle method in number theory, algebra inequalities, elliptical functions etc. He was second Indian to be elected fellow of the Royal Society in 1918. He became first Indian elected fellow of Trinity college, Cambridge. During his illness also he never forget to think about numbers. He remarked the taxi number of Hardy, 1729 is a singularly unexceptional number. It is the smallest positive integer that can be represented in two ways by the sum of two cubes; $\mathbf{1729 = 1^3+12^3 = 9^3+10^3}$. Unfortunately, due to tuberculosis he died in Madras on April 26, 1920. Government of India recognised him and released a postal stamp and declared 2012 as **"Year of Mathematics"** on the eve of his 125th birth anniversary.

SIGNS AND SYMBOLS OF SCHOOL MATHEMATICS

Sign/symbol	Read as	Mathematical meaning
$\pm$	plus or minus	Add or Substract
$\neq$	not equal to	unequal
$\therefore$	therefore	logical flow of a statement
∞	infinite	not finite
$\sim$	is similar to	same in geometrical shape
$\cong$	is congruent to	same shape and same size
$\equiv$	is identically equal to	equivalent statements
$\forall$	for all	universal quantifier
$\sqrt{\ }$	square root of	square root of a number
$\sqrt[3]{\ }$	cube root of	cube root of a number
$\bigcup$	cup of	union of sets
$\bigcap$	cap of	intersection of sets
ϕ	phi	symbol for golden ratio
%	percent of	per hundred
o	degree	angle measure
Δ	delta / triangle	symmetric difference in sets/symbol of triangle
$\in$	belongs to	an element belong to a particular set
$\leftrightarrow$	equivilent to	one to one correspondence
α, β, γ	alfa,beta,gamma	greek lettetrs to represent zeroes of polynomial
μ	mu	universal set symbol
π	pi	circumference of a circle / diameter
σ	sigma	sum of scores
$\sin\theta, \cos\theta, \tan\theta$	sin theta,cos theta,tan theta	trigonometric ratios
$\overline{x}$	x bar	arithmetic mean
$\log_a x$	log x to the base a	logarthemic function
(a, b)	point (a, b)	ordered pair (a, b)
$\lvert x\rvert$	mod x	absolute value of a real number
P(x)	P of x	a polynomial function in x
P(E)	P of E	probability of an event
$\because$	Since	reasoning at a stage
₹	rupee	symbol of Indian rupee
$\parallel$	is parallel to	parallel lines
$\perp$	is perpendicular to	making 90 degree with a line
$\{\ \}$	flower bracket	used to set notation
$\overset{\frown}{PQ}$	arc PQ	arc of a circle
a^2	a square	square of a number
$\angle$	angle	symbol of angle
θ	theta	measurement of an angle

Notes

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